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APPLICATION OF MATHEMATICAL METHODS IN MILITARY AFFAIRS, CHAPTERS II, IV, AND V

I. Anureev, et al

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APPLICATION OF MATHEMATICAL METHODS IN MILITARY AFFAIRS

(CHAPTERS II-IV-V)

I. Anureyev

USSR

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13. ABSTRACT

Some fundamentals of probability theory are laid out. Then, through examples, it is shown how these fundamentals can be used in choosing the means (weapons, units, etc.) for solving certain military problems. Staging aerial attacks and setting up defenses against them is discussed. Problems of deploying different forces (especially aerial) against different enemy targets, using principles of probability so as to obtain the most effective deployment are solved. Various conditions and complications are added in the continuing examples.

Various methods of solving problems which contain many unknown factors (mathematical programming) and linear problems are outlined. Examples are given, concerning, e.g., minimizing transport costs and assigning weapons to targets. Several distribution problems of a military nature are presented. A step-by-step solution for each problem, utilizing methods of linear programming, is given.

The theory of mass service is described. Certain military problems are solved by its use. The problems are of a similar nature to those in the earlier chapters, dealing largely with the distribution of weapons and other equipment.

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CHAPTER II

PRINCIPLES OF APPLICATION OF MATHEMATICAL METHODS IN MILITARY AFFAIRS

The theory of probability is the mathematical science, which studies the principles of chance phenomena.

A chance phenomenom is one that comes out differently each time, during repititions of a single experiment.

Combat operations of troops are typical examples of chance phenomena, since they always come out differently, regardless of how often they are repeated.

The basic conceptions of the theory of probability are: the event, the probability of an event, and chance quantity.

An event is any fact which may happen or not happen as the result of an experiment.

The probability of an event is a numerical measure of the degree of objective possibility of the event occurring.

To compare different events by the degree of the possibility of their occurrence, it is necessary to have a unit of measure. As the unit of measure we use the probability of a certain event, i.e. an event which must occur as a result of the experiment. If to the certain event we assign a probability equal to one, then all other events (possible,

but not certain), will be characterized by a probability of less than one. Opposite to the certain event is the impossible event, the probability of which is equal to zero.

Therefore, the range of changing probabilities is between 0 and 1.

If, in a series of n experiments, the event A appeared m times, then the frequency, or statistical probability is what we call the relationship

$$P^* = \frac{m}{n},$$

where P^* is the frequency of the event,

m is the number of times the event has occurred, and

n is the total of experiments conducted.

In a large number of experiments, the frequency stabilizes, approaching, with very little deviation, a particular constant value, This value, to which the frequency strives in a large number of experiments, is called the probability of the event.

Probability is successfully applied in predicting the results of combat operations, and also for comparing the effectiveness of different combant variants.

Thus, if the probability of a given event in a given experiment is exceedingly small, then one may be practically assured, that in a one-time fullment of the experiment, the expected event will not occur.

If the probability of an event in a given experiment is close to one, then such an event, in practicality, is close to certain.

A random quantity is a quantity which, as the result of an experiment, can acquire one value or another, where it is not known beforehand, what that value will be.

Random quantities can be discrete or continuous. Relating to discrete random quantities are, for instance:

- the number of hits in launching n rockets;

- the number of enemy aerial-space targets knocked down in flight;

- the number of fragments formed by an exploding shell.

Examples of continuous random quantities are:

- the coordinates of points of impact of rockets;

- coordinates of points of interception;

- rocket parameters at the end of the active portion of a trajectory.

One of the most important jobs for the theory of probability is determining the probabilities of events by known probabilities of other events that are related to them. For this, one uses the basic theorems of the theory of probability: the theorem of addition and the theorem of multiplication of probabilities.

The Theorem of Addition of Probabilities

We will introduce several definitions. The sum of two events A_1 and A_2 is the event consisting of the appearance of either of the events. Events are called incompatible when no two of them can occur at the same time.

The theorem of addition of probabilities is formed in the following manner.

The probability of the sum of incompatible events is equal to the sum of the probabilities of the separate events.

$$P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots + P(A_n),$$

where $P(A_1 + A_2 + \ldots + A_n)$ = The probability of the sum of incompatible events (i. e. the appearance of event A_1, A_2, or A_n);

$P(A_1), P(A_2), P(A_n)$ = The probability of separate events.

Two incompatible events are called opposite if one of them must appear in the results of the experiment. An event, opposite to the event A, can be denoted $\overline{A}$.

Examples of opposite events are:

- Hit and miss in rocket firing

- Shooting down a flying device or not.

Since the sum of the events $A + \overline{A}$ is a certain event, we have

$$P(A) + P(\overline{A}) = 1.$$

The Theorem of Multiplication of Probabilities

In preparation, we will introduce two definitions: the product of events, and independent events.

The <u>product</u> of several events is what we call the event that consists of the simultaneous appearance of all these events. Two events are called <u>independent,</u> if the appearance of one of them does not influence the probability of the appearance of the other.

The probability of the product of several independent events is equal to the product of the probabilities of these events:

$$P(A_1, A_2, \ldots, A_n) = P(A_1)\,P(A_2) \ldots P(A_n).$$

In practice, it is most often necessary to apply the theorems of addition and multiplication of probabilities simultaneously. Here, the event, whose probability it is necessary to determine, is presented as the sum of several incompatible events, each of which, in turn, is a product of events. We will examine several examples.

<u>Example One:</u> Three rockets are fired at the same target. The probabilities of a hit in the first, second, and third firing are, respectively, P_1, P_2, P_3.

Find the probability of any one hit.

Solution: We will consider the event A to be equal to one hit on the target. This event breaks down into several incompatible variants: A hit in the first firing, and a miss in the second and third, or, a hit in the second firing and a miss in the first and third, or, a hit in the third and a miss in the first and second.

Consequently:

$$A = A_1\bar{A}_2\bar{A}_3 + \bar{A}_1A_2\bar{A}_3 + \bar{A}_1\bar{A}_2A_3.$$

Applying the theorems of addition and multiplication of probabilities, we obtain:

$$P(A) = P_1(1-P_2)(1-P_3) + (1-P_1)P_2(1-P_3) + \\ + (1-P_1)(1-P_2)P_3.$$

Example Two: An aerial battle occurs between a fighter and a bomber. In the process of the battle, the fighter makes two attacks.

The probability of shooting down the bomber the first time is $P_1^1 = 0.2$. If the bomber is shotdown, it fires back at the fighter and has a probability of shooting it down of $P_2 = 0.3$. If the fighter is not shot down, it continues the attack and has a probability of shooting down the bomber of $P_1^2 = 0.4$. Find the probability of either the bomber or the fighter being shot down.

Solution: We consider the events:

A_1 - bomber shot down,

A_2 - fighter shot down,

The event A_1 can be presented as the sum of two incompatible events:

$$A_1 = A_1^1 + A_1^2,$$

where A_1^1 is the bomber shot down the first time

A_1^2 is the bomber shot down the second time.

The event A_1^2 is the product of three events: non-destruction of the bomber in the first attack, non-destruction of the fighter by the

return fire, and destruction of the bomber in the second attack.

Consequently, on the basis of the theorem of multiplication of probabilities, we have:

$$P(A_1^2) = (1 - P_1^1)(1 - P_2)P_1^2 = 0{,}8 \cdot 0{,}7 \cdot 0{,}4 = 0{,}224.$$

The probability of the bomber being shot down the second time is equal to

$$0.2 + 0.224 = 0.424$$

The event A_2 is the product of two events: non-destruction of the bomber the first time and destruction of the fighter by the return fire. Consequently, the probability of the fighter being shot down is

$$(1 - P_1^1)P_2 = 0{,}8 \cdot 0{,}3 = 0{,}24.$$

Example Three. There are three each of two types of rockets.

First, rockets of the first type are launched, then, rockets of the second type, at the same target. Upon the first hit, the target is destroyed and the rocket launching ceases. The probability of a hit when one rocket of the first type is launched is $P_1 = 0.1$; the probability when one rocket of the second type is launched is $P_2 = 0.2$. Find the probability that not all of the rockets will be spent.

Solution: We will consider the opposite event: $\overline{A}$ - all rockets spent.

This can occur, under the condition that the first five rockets miss the target. Consequently:

$$P(\overline{A}) = (1 - P_1)^3 (1 - P_2)^2 = (1 - 0{,}1)^3 (1 - 0{,}2)^2 = 0{,}466.$$

The probability that all the rockets will be spent is equal to

$$P(A) = 1 - P(\overline{A}) = 1 - 0{,}466 = 0{,}534.$$

The Distribution Laws of Random Quantities

The distribution law of discrete random quantities is what we call the relationship, that establishes the connection between the possible values of a random quantity and their corresponding probabilities.

For the simplest form of the distribution law of a discrete random quantity X, we have a table

X	X_1	X_2	...	X_n
P	P_1	P_2	...	P_n

In the first row are given all the possible values of the random quantity, in the second, the corresponding probabilities of these values appearing.

A knowledge of the laws of distribution allows us to master random quantities to a significant degree. This fact notably eases predicting the outcome of each experiment. Graphically, the distribution law of discrete random quantities takes on the form of a polygon (Fig. 6), which is called the distribution polygon. A distribution polygon completely characterizes a discrete random quantity and is a form of the distributive law.

Example Four: Rockets are launched at several targets until the first hit. The probability of a hit for each launching is equal to P. The random quantity X is the number of launchings. Find the distributive law of the random quantity.

Solution: The possible values of the random quantity are X = 1, 2, 3, ... For the value of the random quantity to be 1, it is necessary that the first launching is a hit; the probability of this value is P.

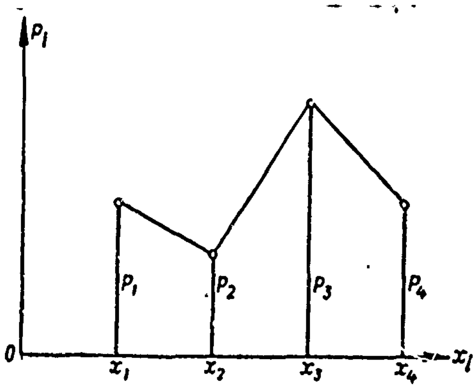

Fig. 6. Distributive Polygon

For the random quantity X to acquire a value of 2, the first launching must be a miss and the second a hit; the probability of this value is equal to (1 - P) P, etc. The distributive law is shown in the form of a table.

X	1	2	3	...	n
P	P	$(1-P)P$	$(1-P)^2P$	...	$(1-P)^{n-1}P$

The distributive law obtained allows one to judge with sufficient certainty the probability of the first rocket hit after a diverse number of launchings. If, for instance, P = 0.2, then the probability of the second rocket being the first hit will be 0.16, and the probability of the second rocket being the first hit will be 0.16, and the probability of the fourth rocket being the first hit is already 0.1, or two times less, than for the first rocket.

Of important significance in military affairs are the binominal distribution law and Poisson's Law.

The binominal distribution law establishes the dependence of the probability of an event occurring k times in n independent experiments. In application to military weapons, this law can be used to determine the number of hits for n independent firings at the same target.

The probability of hitting, for k weapons in n independent launchings, is determined by the formula

$$P_n(k) = \frac{n(n-1)(n-2)\dots(n-k+1)}{1\cdot 2\dots k} P^k (1-P)^{n-k},$$

where $P_n(k)$ is the probability of a hit by k weapons with n independent firings at the same target; and

P is the probability of a hit in one firing.

The random quantity, in this law, takes the discrete values: 0, 1, 2, n. The distribution law can be written in the form of a table.

k	0	1	2	...	$n-1$	n
$P_n(k)$	$(1-P)^n$	$nP(1-P)^{n-1}$	$\frac{n(n-1)}{2}P^2(1-P)^{n-2}$	...	$nP^{n-1}(1-P)$	P^n

For instance, if five rockets are launched at the same enemy starting position, and the probability of hitting the target with one rocket is 0.8, then the probability of missing with all the rockets (k = 0), and in the same way. the probabilities of hitting with one rocket, two two rockets, etc., will have values entered on the table.

K	0	1	2	3	4	5
P_k	0,0003	0,0064	0,05	0,21	0,41	0,33

In checking the example, we find that the probability of missing with all the rockets is considerably smaller than the probability of hitting with all the rockets. The probabilities of hitting with either four or five rockets are the same and have the largest value.

Poisson's Law expresses the dependence of the probabilities of an event occurring k times in n independent tests, where the probability of each event is the same. This L.w is good for mass events (n is large) and rare events (the probability of the event occurring is small.)

Poisson's Law is expressed by the formula

$$P_n(k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

where $P_N(k)$ is the probability of an event occurring k times in n independent experiments;

λ = nP, where n is the number of experiments and P is the probability of the event occurring in one experiment; and

! is the factorial sign.

Giving k integral values, Poisson's distribution law can be written in the form of a table.

k	0	1	2	...	k
$P_n(k)$	$e^{-\lambda}$	$\lambda e^{-\lambda}$	$\frac{\lambda^2}{2}e^{-\lambda}$	...	$\frac{\lambda^k e^{-\lambda}}{k!}$.

Applying Poisson's Law will be examined below in analyzing the theory of mass service.

Mathematical Expectancy

The mathematical expectancy of a discrete random quantity is what we call the sum of the products of all of its (the random quantity's) possible values by their probabilities:

$$M = X_1P_1 + X_2P_2 + \ldots + X_nP_n = \sum_{i=1}^{n} X_iP_i,$$

where M is the mathematical expectancy,

X_1, X_2, ... X_n are the values of the random quantity,

and P_1, P_2, ... P_n are the probabilities of each value of the random quantity.

Although the mathematical expectancy tells one considerably less about a random quantity, than its distribution law, it is, nevertheless, an exceedingly important characteristic of the random quantity.

Example Five: Three rockets are fired at the same target, and the probabilities of hitting it are, respectively, $P_1 = 0.5$, $P_2 = 0.4$, $P_3 = 0.6$. Find the mathematical expectancy of the total number of hits.

Solution: Applying the formula for mathematical expectancy, we obtain:

$$M = X_1P_1 + X_2P_2 + X_3P_3 = 1 \cdot 0{,}5 + 1 \cdot 0{,}4 + 1 \cdot 0{,}6 = 1{,}5.$$

Example Six: 40 fighters participate in an aerial battle against bombers and have a probability of shooting one down in one attack of $P_f = 0.6$, Also participating are 20 guided anti-aircraft rockets with the probability of a hit in one launching of $P_r = 0.8$. The fighters and the anti-aircraft rockets lead one attack each on different bombers. Determine the mathematical expectancy of the number of bombers shot down.

Solution: The number of bombers shot down is a random quantity. The mathematical expectancy of this quantity is equal to

$$M = 40 \cdot 0{,}6 + 20 \cdot 0{,}8 = 40.$$

Dispersion

In practice, it is very important to appreciate the scattering of possible values of a random quantity around its mathematical expectation. For instance, in rocket launching, it is necessary to know how closely the rockets will fall near the center of dispersion (the point of average miss).

By dispersion of a random discrete quantity, we mean the sum of the products of the squares of the deviations of a random quantity from its mathematical expectation, and the probabilities of these values.

$$D = (X_1 - M)^2 P_1 + (X_2 - M)^2 P_2 + \ldots + (X_n - M)^2 P_n,$$

where D is the dispersion of the random quantity

$X_1, X_2, \ldots, X_n$ are the values of the random quantity.

$P_1, P_2, \ldots, P_n$ are the probabilities of the values of the random quantity, and

M is the mathematical expectation of the random quantity.

Average Quadratic Deviation

Most frequently, the dispersion of a random quantity is characterized by the average quadratic deviation, which has the dimensionality of a random quantity.

By the average quadratic deviation of a random quantity we mean the square root of the dispersion.

$$\sigma = \sqrt{D} = \sqrt{(X_1 - M)^2 P_1 + (X_2 - M)^2 P_2 + \ldots + (X_n - M)^2 P_n},$$

where σ is the average quadratic deviation.

Example Seven: Ten ballistic rockets of the first type are launched over the same distance. In four launchings the rockets deviated by 1 km, in one launching by 2 km, and in five, by 3 km.

Ten ballistic rockets of a second type were also launched over the same distance. In one launching a rocket deviated by 1 km, in six launchings by 2 km, and in three launchings, by 3 km. Determine the mathematical expectation, dispersion, and the average quadratic deviation of misses for both types of rockets.

Solution: Accepting that the probability of an event is proportionate to its frequency, one may obtain the laws of distribution of a random quantity, of the deviation (misses) of rockets of both types. These distributive laws can be shown in the form of a table.

a Значения промаха, км		1	2	3
b Вероятности появления этих значений	c первый тип ракеты	0,4	0,1	0,5
	d второй тип ракеты	0,1	0,6	0,3

Key: a. Value of miss, km; b. probability of these values occurring; c. First type of rocket; d. second type of

The mathematical expectation of the magnitude of a miss is equal to

$$M_{I} = 1\cdot 0{,}4 + 2\cdot 0{,}1 + 3\cdot 0{,}5 = 2{,}1 \text{ км};$$
$$M_{II} = 1\cdot 0{,}1 + 2\cdot 0{,}6 + 3\cdot 0{,}3 = 2{,}2 \text{ км}.$$

Comparing the mathematical expectations of the size of a miss, we can say, that in many launchings of rockets of the first type, we will get somewhat better results in accuracy. However, a more detailed examination of the other numerical characteristics of the random quantity will allow one to bring out the positive qualities of the second type of rocket.

The dispersion of misses will be equal to

$$D_{I} = (1-2{,}1)^2 0{,}4 + (2-2{,}1)^2 0{,}1 + (3-2{,}1)^2 0{,}5 = 0{,}89 \text{ км}^2;$$
$$D_{II} = (1-2{,}2)^2 0{,}1 + (2-2{,}2)^2 0{,}6 + (3-2{,}2)^2 0{,}3 = 0{,}337 \text{ км}^2.$$

The average quadratic miss deviation forms

$$\sigma_{I} = 0{,}943 \text{ км};$$
$$\sigma_{II} = 0{,}58 \text{ км}.$$

From here, it is apparent, that impact points for rockets of the second type are less dispersed, more closely grouped around the average value (and this means among themselves), than rockets of the first type. Consequently, in single launchings of rockets of the second type, the deviation of the factual point of impact from the average miss value will be about 1. 6 times less than that of rockets of the first type. And insofar as the average miss value of both rockets is approximately the same, one should prefer the seoond type of rocket for single launchings.

Distributive Functions

A continuous random quantity has an infinite set of possible values, which entirely occupy a certain interval. Therefore, the

probability that a continuous random quantity will take a certain fixed value is equal to zero. For a quantitative valuation of the distribution of probabilities of a continuous random quantity, one uses the probability of the inequality $P(X < x)$, where x is a continuous variable.

The probability of the inequality $P(X < x)$ signifies the probability that the random quantity X will acquire a value less than a certain assigned value x:

$$F(x) = P(X < x).$$

The probability of the introduced inequality depends on x and is called the distribution function. Geometrically, the distribution function can be interpreted thus: F (x) is the probability that the continuous random quantity will acquire one of the values on the portion left of point x (Fig. 7)

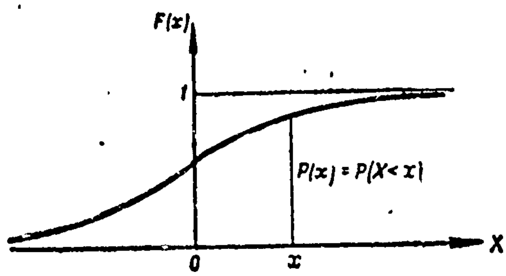

Fig. 7. Distribution Function

The distribution function is also called the integral distribution law.

The basic properties of the distribution function are as follows:

-- The function F (x) is the non-decreasing function of its argument;
-- At minus infinity the distribution function is equal to zero:

$$F(-\infty) = 0;$$

-- At plus infinity the distribution function is equal to one:

$$F(+\infty) = 1.$$

In solving practical problems, it is necessary to determine the probability that the random quantity will acquire a value contained in certain boundaries from α to β. Geometrically, this means that a point whose coordinate is equal to the random quantity will fall on the segment $\alpha\beta$. (Fig. 8).

A large number of problems of a military character, for instance, problems in destroying enemy supply objectives, belong to this type of problem.

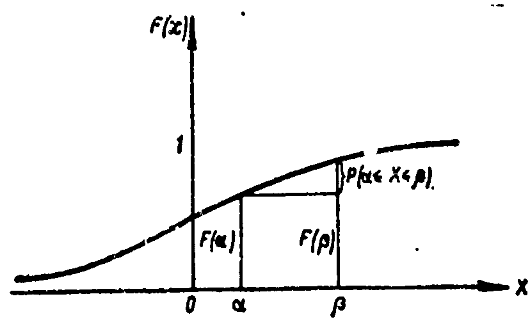

Fig. 8. The Probability of a Chance Quantity Falling on a Segment.

The probability of the random quantity falling on the segment is equal to the increase of the distribution function in this segment:

$$P(\alpha \leqslant X \leqslant \beta) = F(\beta) - F(\alpha),$$

where $P(\alpha \leqslant X \leqslant \beta)$ is the probability of the random quantity falling on the segment, $\alpha\beta$;

$F(\alpha)$, $F(\beta)$ are the values of the distribution function at points α and β.

Distribution Density

The distribution density (or differential distribution function) is the limit of the ratio of the probability of a random quantity falling on

segment to the length of that segment, as the length of the segment approaches zero.

$$f(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x},$$

where f(x) - is the distribution density, and

F(x) is the distribution function.

Consequently, we can write

$$f(x) = F'(x) = \frac{dF}{dx},$$

i.e. the distribution density is equal to the derivative of the distribution function (Fig. 9)

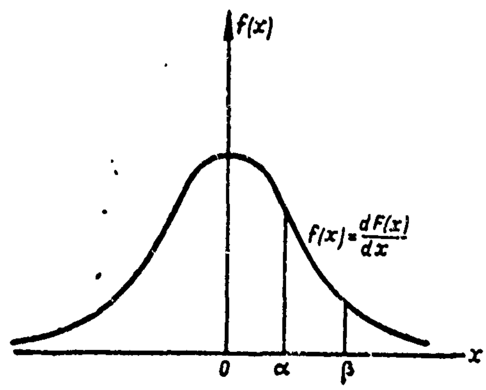

Fig. 9. Distribution Density

From this relationship one can determine the probability of the random quantity falling on the segment $\alpha\beta$.

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f(x)\,dx.$$

Normal Distribution Law

The normal distribution law (Gauss' Law) is the most frequently met in solving military problems. All theoretical questions of firing, bombing, intelligence effectiveness, radio counter-action, and reliability of technical systems are based on use of the normal distribution law. This law is the limiting law, to which all other distribution laws approach in frequent, typical conditions.

The normal distribution law is characterized by a probability density:

$$f(x) = \frac{\rho}{E\sqrt{\pi}} e^{-\frac{\rho^2}{E^2}x^2},$$

where f(x) is the probability density,

ρ = 0.477 is the constant of the normal distribution law; and

E is the probable deviation.

The probable deviation (or average deviation) is half the length of the segment, into which the probability of falling is 50%.

The probable deviation for the normal is connected to the average quadratic deviation by the dependency

$$E = \rho\sqrt{2}\,\sigma = 0{,}674\sigma,$$

where σ is the average wuadratic deviation.

The Probability of a Random Quantity Falling on a Segment

The probability of a random quantity, subject to the normal law, falling on a segment is equal to the difference of the values of the distribution function. The distribution function for the normal law carries the name, the Laplace Function (the positive region of determination).

In this manner, we have

$$P(\alpha < X < \beta) = \frac{1}{2}\left[\hat{\Phi}\left(\frac{\beta}{E}\right) - \hat{\Phi}\left(\frac{\alpha}{E}\right)\right],$$

where $\hat{\Phi}(x)$ is the Laplace function reduced.

Certain values of the reduced Laplace function are given in Table 2.

TABLE 2

x	$\hat{\Phi}(x)$	x	$\hat{\Phi}(x)$	x	$\hat{\Phi}(x)$
0	0	0,5	0,264	3,0	0,957
0,1	0,054	1,0	0,500	4,0	0,993
0,2	0,107	1,5	0,688	4,5	0,998
0,3	0,160	2,0	0,823	5,0	0,999
0,4	0,213	2,5	0,908		

If the length of the segment is equal to 1 (one), and the center of dispersion coincides with the center of the segment, then the probability of the random quantity falling on the segment, that is, on an infinite band of width 1 (one), equal to

$$P = \hat{\Phi}\left(\frac{l}{2E}\right).$$

By this formula, with the aid of Table 2, one may calculate the probabilities of falling in consecutive segments of length E, lying away from the center of dispersion, which are reduced in the table below

a Расстояние от центра рассеивания в долях E	E	$2E$	$3E$	$4E$
b Вероятность, %	0,25	0,16	0,07	0,02

Key: a. Distance from center of dispersion in units E;
b. Probability.

In this manner, a chance quantity, subject to the normal law, deviates, in practicality, not more than four deviations from the center of dispersion.

The Probability of Falling in a Rectangle

If the center of dispersion coincides with the center of a rectangle (Fig. 10), then the probability of falling in it is:

$$P = \hat{\Phi}\left(\frac{a}{2E_x}\right)\hat{\Phi}\left(\frac{b}{2E_y}\right),$$

where P is the probability of falling in the rectangle,

a, b are the dimensions of the rectangle, and

E_x, E_y are the probable deviations in the directions of X and Y axes (the axes parallel to the sides of the rectangle).

Example Eight: Rockets are fired at an airstrip whose dimensions are 2000 by 80 m.[2] The dispersion of the rockets is circular with a probable deviation of 150 m. Determine the probability that the rocket will fall on the strip.

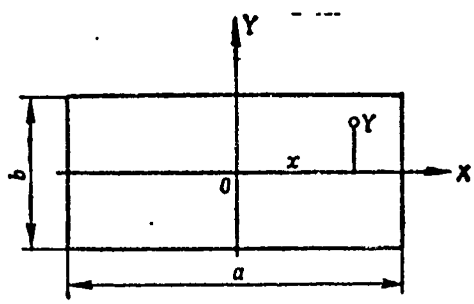

Fig. 10. The Probability of a Random Quantity Falling in a Rectangle

Solution: Substituting the figures into the formula for probability of falling in a rectangle, we obtain

$$P = \hat{\Phi}\left(\frac{2000}{2\cdot 150}\right)\hat{\Phi}\left(\frac{80}{2\cdot 150}\right) \approx \hat{\Phi}(6{,}7)\,\hat{\Phi}(0{,}27) \approx 1\cdot 0{,}15 \approx 0{,}15.$$

The Probability of Falling in a Dispersion Ellipse

Often it is necessary to consider a system of random quantities, distributed in space or on a plane. In the case of two independent random quantities we may speak of the domain of their definition on a plane.

If the random quantities are subject to the normal distribution law, then all the possible values of two independent random quantities are dispersed so that they will form an ellipse on a plane, which is called an ellipse of dispersion.

Usually, two types of ellipse are considered, the unit ellipse and the full ellipse.

By unit ellipse of dispersion, we mean an ellipse that is equal to the probability density, whose semiaxis is equal to the principle probable deviation. Its equation takes the form

$$\frac{x^2}{E^2} + \frac{y^2}{E^2} = 1.$$

By full ellipse of dispersion, we mean an ellipse equal to the probability density, whose semiaxis is equal to quadruple the principle probable deviations. Its formula is:

$$\frac{x^2}{(4E)^2} + \frac{y^2}{(4E)^2} = 1.$$

The full ellipse of dispersion contains practically all of the dispersions on a plane, since the probability of a random point falling outside of its borders is small (about 0. 03). If the principle probable deviations are equal to each other, then the ellipse of dispersion turns into a circle and the dispersion is called circular. Circular dispersion occurs in bombing and in firing of many types of rockets.

The probability of falling in an ellipse of dispersion, whose semiaxis is equal to λ probable deviations, is expressed by the formula

$$P = 1 - e^{-\lambda^2\rho^2}.$$

The probabilities of falling into unit and full ellipses are equal, correspondingly, to

$$P = 1 - e^{-\rho^2} = 1 - e^{-0{,}477^2} = 0{,}203;$$

$$P_{\text{полн}} = 1 - e^{-(4\rho)^2} = 0{,}974.$$

As a characteristic of circular dispersion, radial probable deviation is often applied.

Radial probable deviation is what we call the radius of a circle of dispersion, in which the probability of falling is 50%. Radial deviation is connected to probable deviation by the relationship

$$E_{\text{рад}} = 1{,}75E.$$

Mathematical Expectation of Damage

The mathematical expectation of damage inflicted on the enemy as a result of combat operations objectively characterizes the combat capacity of the armament and presents an average number of destroyed enemy units, an average reduction in the productivity of the objective, the average destruction of separate elements of the target, etc.

In considering the military application of flying devices, (ballistic and winged rockets, bombers) against ground targets, we may apply, as a criterion of combat effectiveness, the mathematical expectation of the relative damage inflicted on a flat target.

At first, we will take into account the enemy's counter action and the technical reliability of our flying devices.

Suppose armaments are thrown at a flat-surface target, which will destroy the target within their radius.

We may assume, that on the area of the target there will be placed another area -- the region of disruption. In rocket launching or bombing one is striving to make the center of the target area and the center of the area of disruption coincide. However, as a result of dispersion, there

is a random deviation of the center of the area of disruption from the center of the target area. These deviations are subject to the normal law.

For the real contours of the target and the area of disruption we will substitute equal rectangles $\overline{T}_x\overline{T}_y$ and $\overline{L}_x\overline{L}_y$ (Fig. 11).

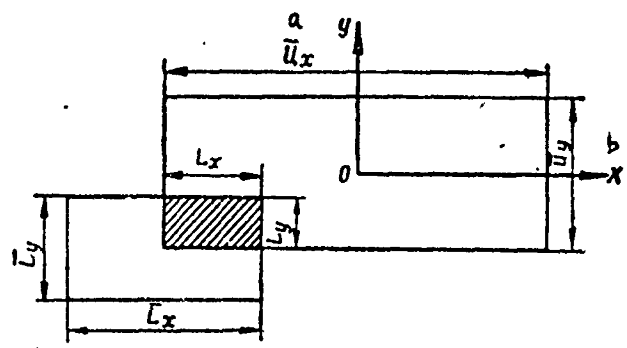

Fig. 11. To a Determination of the Mathematical Expectation of Relative Damage

Key: a. T_x; b. T_y.

The relative area overlapped by the area of disruption and the target area, i. e., the area of destruction, is equal to:

$$M=\frac{L_xL_y}{\overline{u}_x\overline{u}_y}=M_xM_y,$$

Key: a. $\overline{T}_x\overline{T}_y$

where M is the relative damage;

L_xL_y are the lengths of the overlap area in the directions of x and y,

T_x and T_y are the dimensions of the target, and

M_x and M_y are the relative lengths of overlap in directions x and y.

The mathematical expectation of the relative damage inflicted on a flat-surface target is determined by the formula

$$M = M_x M_y$$

Since the formula for finding M_x and M_y has an unwieldy form, in practicality, problems of determining the degree of destruction by arms are solved with the help of electronic computers, special tables, graphics and nomograms.

For stationary electronic calculators, algorithms and programs have been developed, which allow a quick determination of the degrees of destruction of objectives by armaments.

Very good results are given by the use of tables, graphics, and nomograms, made from calculations by electronic computers for typical enemy objectives and basic military weapons.

If the dimensions of the target are large by comparison to the radius of destruction and the dispersion is circular, then, for approximate calculations, one may determine the mathematical expectation of the relative damage by the formula

$$M = \left(\frac{\overset{a}{R_{\text{п}}}}{\underset{b}{R_{\text{ц}}}}\right)^2 \left[1 - e^{-\rho^2\left(\frac{\overset{b}{R_{\text{ц}}} - \overset{a}{R_{\text{п}}}}{E}\right)}\right]^2,$$

Key: a. R_p ; b. R_t

Where M is the mathematical expectation of the relative damage;

R_p is the radius of destruction;

R_t is the radius of the target (the radius of a circle equal in area to the target);

E is the probable deviation.

Probability of Destruction of a Target

Destruction of a target means that through the use of weapons the functioning of one or more elements has been destroyed, as a result of

which, the target ceases to fulfill its tasks.

The probability of destroying a target is the probability of inflicting so much damage, that the target goes completely out of order and its continued functioning is terminated.

The probability of destroying a target depends on the destructive power of the weapons being used, the quantity of weapons fired at the target, the dispersion characteristics of carriers of nuclear weapons, and the viability of the target itself.

The probability of destroying a target is identified with the probability of the weapons hitting in a circle, whose radius is equal to the radius of target destruction. With circular dispersion the probability of target destruction is determined by the formula

$$W = 1 - e^{-\rho^2 \frac{R_n^2}{E^2}},$$

where W is the probability of target destruction,

$\rho = 0.477$ is the constant for the normal law,

R_n is the radius of destruction, and

E is the probable deviation.

We identify the ratio of the radius of destruction to the probable deviation

$$\frac{R_n}{E} = K.$$

In such an instance, the probability of destruction of a target is written in the form

$$W = 1 - e^{-\rho^2 K^2}.$$

In Table 3 are entered probabilities of target destruction in percentages, depending on quantities K.

TABLE 3

	K	b Десятые доли									
		0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
a Целые доли	0	0	0,2	1	2	4	6	8	11	14	17
	1	20	24	28	32	36	40	44	48	52	56
	2	60	63	67	70	73	76	79	81	83	85
	3	87	89	90	92	93	94	95	96	96	97
	4	97	98	98	99	99	99	99	99	99	100
	5	100	100	100	100	100	100	100	100	100	100

Key: a. Tenths; b. Whole numbers

Example Nine: Determine the probability of destroying a launch position of the open type with a ballistic rocket. The radius of destruction is 2.25 km, and the rocket's probable deviation is 1.5 km.

Solution. We calculate the quantity K: $K = \frac{R_n}{E} = \frac{2,25}{1,5} = 1,5.$

From Table 3 we derive the probability of destroying the target:

$$W = 40\%.$$

Example Ten: Determine the probability of destroying a starting position of the pit type with a ballistic rocket. The radius of destruction is equal t .4 km, the probable deviation of the rocket - to 1.5 km.

Solution: We calculate the quantity K

$$K = \frac{1,4}{1,5} = 0,93.$$

From table 3 we find the probability of destroying the pit type starting position:

$$W = 17\%.$$

Example Eleven: Fighter-bombers inflict a blow on an enemy guided-missile battery. The radius of destruction is equal to 1.5 km, the probable deviation of the weapons is 0.1 km. Determine the probability of destruction.

Solution: We calculate the quantity K:

$$K = \frac{1,5}{0,1} = 15.$$

From Table 3, we determine the probability of destroying the battery.

$$W = 100\%.$$

One should note, that if K is greater than 5, the probability is equal to 100%.

Evaluating the Effectiveness of Combat Operations by Calculating the Enemy's Counter-Action.

In analyzing combat operations, the most important and complex task is calculating the enemy's counteraction.

In evaluating the enemy's counter-action, we may distinguish two basic cases:

-- When the enemy's counter-action precedes the operation being considered, whose effectiveness is being evaluated;

-- When the enemy counter-action occurs in the course of the military task.

The first case is more simple from the mathematical point of view and, at the same time, can be met sufficiently often in military circumstances.

For instance, in determining the effectiveness of strikes by aviation, we must count the losses, which the enemy inflicts upon us in the air and on the ground. In this connection, only the planes which reach the assigned objective will carry out a strike against it.

In the case under consideration, the enemy counter-action precedes our operation.

The scheme for calculating the enemy's counter-action, preceding the fulfillment of a combat mission, depends on how many units fulfill

the mission. First, we will consider the case, where the mission is fulfilled by one military unit (rocket, plane, artillery, battery, tank or subdivision of motorized infantry).

As the criteria of effectiveness we take the probability of fulfilling a combat mission W or the mathematical expectation M of damage inflicted on the enemy. We indicate by W_d the probability that the military unit will not be destroyed by the enemy's counteraction. From here on, this quantity will be called the probability of reaching the target. Obviously, for the military unit to fulfill the assigned mission, it is necessary, first of all, that it not be destroyed.

Consequently, in order to calculate the preceding enemy counteraction, it is necessary to multiply the criterion of effectiveness, calculated without considering the counteraction, by the probability of reaching the target.

Indicating the criterion of effectiveness with consideration of counteraction by W_{pr} or M_{pr}, we obtain

$$W_{pr} = WW_D \quad or \quad M_{pr} = MW_D$$

Example Twelve: Our tank attacks an enemy tank located in concealment. The enemy tank begins firing earlier than ours. The probability of our tank being destroyed is $W_o = 0.5$, and the probability of destroying the enemy tank is $W_e = 0.4$. Determine the criterium of effectiveness of our tank with consideration of the enemy's counteraction (return fire).

Solution. Applying the formula for the probability of destruction with consideration of counteraction, we obtain

$$W_{pr} = W_p W_D = 0{,}4\,(1 - 0{,}5) = 0{,}2.$$

Example Thirteen: A bomber makes a strike against a troop concentration. The probability of the bomber being shot down by the enemy anti-aircraft weapons is $W_{aa} = 0.2$. The bomber's effectiveness is 0.7,

relative to the destruction it inflicts on the enemy troop concentration. Determine the bombers criterion of effectiveness, taking into account the enemy anti-aircraft counteraction.

Solution: Applying the formula for mathematical expectation of damage with counteraction, we obtain

$$M_{\text{рф}} = MW_{\text{д}} = 0{,}7\,(1-0{,}2) = 0{,}56.$$

We will consider the case where several military units perform a mission (tank group, infantry unit, rockets, planes).

A group of military units, used as a means of enemy counteraction, is a group target.

For simplicity, we will consider that the military units, making up the group target, are subject to counteraction, independently of one another. We indicate by W_1, W_2, $W_3, \dots, W_n$ the probabilities of destroying the enemy objective with each separate unit, and by W_{1d}, W_{2d}, $\dots$, W_{nd} the probabilities that these military units will reach the target (the probabilities of their not being destroyed).

Then the probability of fulfilling a mission with a group of military units with enemy counteraction is expressed by the formula

$$W_{\text{пр}} = 1-(1-W_{1\text{д}}W_1)(1-W_{2\text{д}}W_2)\dots(1-W_{n\text{д}}W_n).$$

Key: a. pr; b. d.

If the probabilities of destroying the enemy objective by the military units and the probabilities of their reaching the target are correspondingly the same, then the preceding formula takes on a simpler form.

$$W_{\text{пр}} = 1-(1-W_{\text{д}}W)^n,$$

pr d

Where W_{pr} is the probability of destroying the target with a group of military units.

W_d is the probability of reaching the target for each military unit.

W is the probability of destroying the target for each military unit, and

n is the number of identical military units.

Therefore, if the military units are subjected to enemy counteraction independently of one another, then to calculate the enemy counteraction it is sufficient to multiply the criterion of effectiveness of each unit by the probability of reaching the target, and then use the formula with the exponential law.

If, as a criterion of effectiveness, we take the mathematical expectation of damage, then the formula for the mathematical expectation of damage with enemy counteraction takes the form

$$M_{np} = 1 - (1 - W_{1д}^{b} M_1)(1 - W_{2д}^{b} M_2) \dots (1 - W_{nд}^{b} M_n);$$
$$M_{np}^{a} = 1 - (1 - W_{д}^{b} M)^n.$$

Key: a = pr; b = d

<u>Example Fourteen:</u> A strike is made by a chain of fighter-bombers with the usual weapons against an enemy launching pad for tactical-operational rockets. The probability of destroying the launching pad with one fighter-bomber is equal to $W_{fb} = 0.4$. The probability of a fighter-bomber being destroyed by the enemy anti-aircraft fire is $W_{db} = 0.25$. There are 4 fighter-bombers in the chain. Determine the probability of destroying the launching pad with the chain of fighter-bombers with consideration of enemy counteraction.

Solution. Applying the formula for unconditional probability of destruction by identical military units, we obtain

$$W_{np}^{a} = 1 - (1 - W_{д}^{b} W)^n = 1 - [1 - (1 - W_{cб}) W_{и.б}]^n =$$
$$= 1 - [1 - (1 - 0{,}25)\, 0{,}4]^4 = 1 - 0{,}7^4 = 0{,}76.$$

Key: a. pr; b. d; c. sb.; d. ib.

Earlier, we considered a method of calculating enemy counteraction, when the courteraction precedes the fulfillment of the combat mission.

In practice, in a number of cases, the enemy counteraction does not precede the military mission, but occurs in the course of the operation. This is observed especially often in conducting operations with ground forces, and most of all with motorized infantry and tank units and combinations. The case that is most simple for calculating counteraction in the course of operations is where the consecutive moments of fire are known and considered to be given. A scheme for calculating counteraction, based on such an assumption, is called the scheme of consecutive strikes of the discrete scheme.

However, one should notice, that when a large number of combat devices are participating in an operation, mathematical analysis by the discrete system becomes exceedingly complex, demanding the use of an electronic calculator.

To illustrate this method, we will consider a battle between two tanks. We will set the following battle scheme. Our tank attacks first from a defined distance with a probability of W_1^1 (the upper index is the number of the shot). If the enemy tank is not destroyed, it will fire at our tank and destroy it with a probability of W_2^1. If our tank is not destroyed, it will continue the attack, fire a second time at the enemy tank and destroy it with a probability of W_1^2. If the enemy tank is not destroyed by this, it fires again at our tank and destroys it with a probability of W_2^2. It is necessary to analyze the firing effectiveness of each tank and find the characteristics of effectiveness.

We will consider the different possible outcomes of the tank battle.

$A_{1,1}$ - neither tank destroyed

$A_{1,0}$ - our tank whole, enemy tank destroyed;

$A_{0,1}$ - our tank destroyed, enemy tank whole.

Using the theorems of addition and multiplication of probabilities, we obtain the probabilities of the possible outcomes of the battle.

$$W_{1,1}=(1-W_1^1)(1-W_2^1)(1-W_1^2)(1-W_2^2);$$
$$W_{1,0}=W_1^1+(1-W_1^1)(1-W_2^1)W_1^2;$$
$$W_{0,1}=(1-W_1^1)W_2^1+(1-W_1^1)(1-W_2^1)(1-W_1^2)W_2^2.$$

Where $W_{1,1}$ is the probability of neither tank being destroyed in two attacks,

$W_{0,1}$ is the probability of our tank being destroyed in two attacks

$W_{1,0}$ is the probability of the enemy tank destroyed in two attacks.

From the example we have studied, we can clarify the method of calculating counteraction by the discrete scheme of battle, and also the complications which occur in the case of a battle of a large number of units.

Fuller and more useful results for calculating enemy counteraction may be obtained by doing "models" of combat operations on electronic computers.

Computing the Array of Weapons Needed to Complete a Military Mission

Choosing the array of military weapons is a most important problem, which must be solved in organizing combat operations of units, sub-units, combined forces, in planning operations, and in the course of military operations.

A complete solution of this problem is difficult without using modern mathematical methods. For a large number of weapons, brought into the operation, computations become complex, and at the present time they are more and more frequently solved on computers.

We will examine a general scheme for solving problems in the needed array of military weapons.

Suppose we have to solve a combat mission (destroy a group of enemy rocket forces, a group of tanks, a troop concentration, etc.) and the effectiveness of its completion is judged by certain criterion of effectiveness M or W.

It is necessary to determine the quantity of weapons (rockets, planes, tanks, etc.) for which the criterion of effectiveness reaches an assigned value M_3 or W_3.

In the capacity of a criterion of effectiveness for fulfilling a given mission with given weapons, one may take:

-- The probability of destroying a single target;

-- the mathematical expectation of the number of destroyed weapons in the composition of a group target;

-- The probability of destroying not less than an assigned portion of the weapons composing a group target.

-- The probability of destroying not less than a given portion of the area of the target.

For a single target, the problem is formulated in the following manner; how many weapons (rockets, planes, etc.), are needed a target with a given probability?

Suppose n independent launchings are made against a target, and the probability of destruction is the same for each. Then the probability of destruction will be euql to

$$W_n = 1 - (1 - W)^n,$$

Where W_n is the probability of destroying the target in n launchings;

W is the probability of destroying the target in one launching.

Equalizing W_n with the assigned value of the criterion W_3 and solving the equation relative to n, we obtain

$$n = \frac{\lg(1 - W_3)}{\lg(1 - W)}.$$

The formula obtained is the basic one in computing the order of forces in independent launchings.

Below is Table 4, in which for certain values of the probability of destruction and for certain magnitudes of the probability of destroying the target with one means, are given values of the number of weapons needed, for independent launchings.

TABLE 4

M_3, W_3, %	M. W. %				
	10	20	40	60	80
50	7	3	1	1	1
60	9	4	1	1	1
70	12	5	2	1	1
80	15	7	3	2	1
90	22	10	4	3	2
95	29	13	6	4	2

For practical calculations, it is convenient to use a graph instead of a table (Fig. 12).

Example Fifteen: With one volley of non-guided reactive shells an enemy command point is destroyed with a probability of W = 0. 2. How many volleys must be fired to destroy the command point with a probability of W_3 = 0. 8?

Solution: By the formula we have

$$n = \frac{\lg(1-0{,}8)}{\lg(1-0{,}2)} = 7.$$

The same result is obtained from Table 4.

Example Sixteen: An enemy starting position is destroyed by a rocket with a probability of 0. 4. How many rockets are needed to destroy the starting position with a probability of W_3 = 0. 95?

Solution.

$$n = \frac{\lg(1-0{,}95)}{\lg(1-0{,}4)} = 6.$$

Example Seventeen: An anti-aircraft rocket shoots down an enemy plane with a probability of W = 0. 6. Determine the number of rockets necessary to destroy an enemy plane with a probability of W_3 = 0. 9.

Solution:

$$n = \frac{\lg(1-0{,}9)}{\lg(1-0{,}6)} = 3.$$

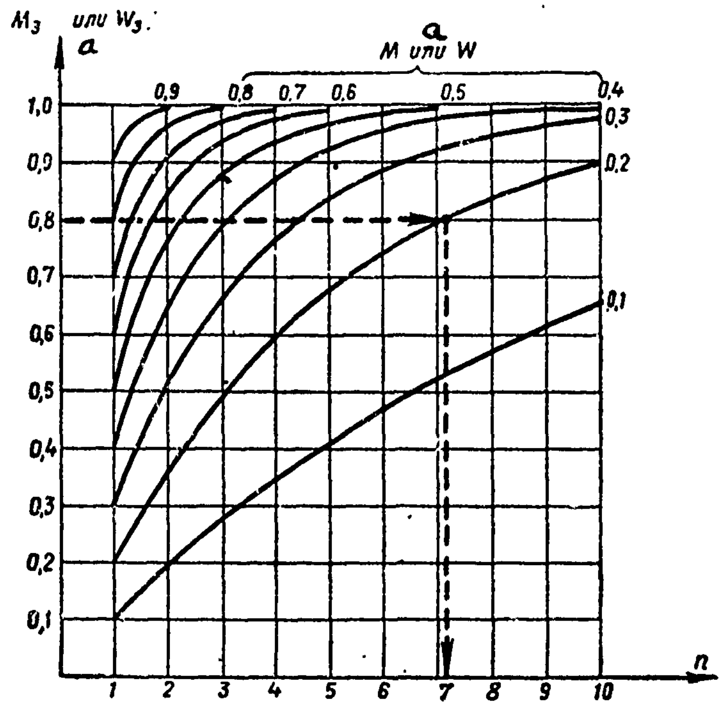

Fig. 12. For determining Weapons Needed to Secure Target Destruction at an Assigned Level
Key: a. or

If, as the criterion of effectiveness, we take the mathematical expectation relative to damage inflicted on a flat-surface target, then the needed amount of weapons may be determined by the formula

Where n is the average number of weapons;

M_3 is the assigned level of destruction of a flat-surface target, and

M is the mathematical expectation of the damage, inflicted by one weapon.

In this way, to determine the needed amount of weapons, one must, first of all, be able to find the mathematical expectation of the damage inflicted by one weapon.

For this there are many graphs and tables. We will enter one of the simplest graphs, allowing the determination of the mathematical expectation of damage in the application of special weapons.

The order in which a problem is solved in determining the average value of the needed quantity of weapons with the use of the graph (Fig. 13), will be as follows:

1. We define the relative radius of target destruction by the special weapon in units of the probable deviation.

$$R_x = \frac{R_p}{E} \text{ и } R_y = \frac{R_p}{E},$$

where R is the relative radius of target destruction;

R_p is the radius of target destruction for the special weapon,

E is the probable deviation in launching the special weapon.

2. We calculate the relative dimensions of the target in units of probable deviation (we see the target approximately as a rectangle or square):

a

$$Ц_x = \frac{\overline{Ц}_x}{E}, \quad Ц_y = \frac{\overline{Ц}_y}{E},$$

Key: a. T

Where T_x, T_y are the relative dimensions of the target;

$\overline{T}_x$, $\overline{T}_y$ are the linear dimensions of the target

E is the probable deviation of circular dispersion.[1]

[1] In the case of non-circular dispersion one must find the quantities separately

$$R_x = \frac{R_p}{E_x}, \quad R_y = \frac{R_p}{E_y}.$$

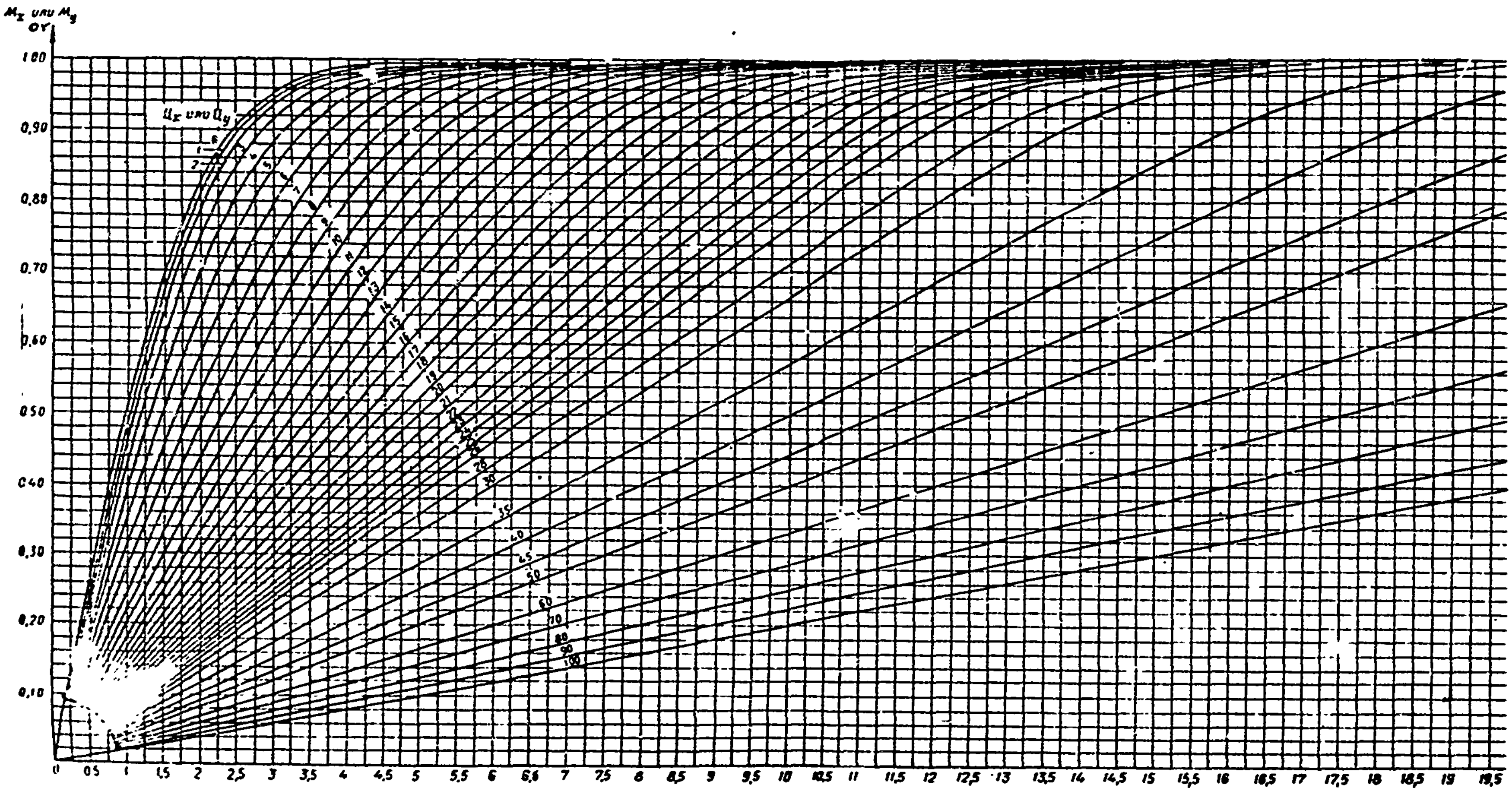

Fig. 13. To determine the mathematical expectation of destruction of flat-surface Target.

3. By the input parameters R, T_x, T_y we find on the graph the relative linear overlapping of the surface of the target by the area of disruption: M_x and M_y.

4. We determine the mathematical expectation of the damage inflicted on the flat-surface target by one weapon:

$$M = M_x M_y.$$

5. Determine the average amount of firing-range weaponry, needed to invlict a given amount of damage M_3 on a target:

$$n = \frac{\lg(1 - M_3)}{\lg(1 - M)}.$$

6. In considering the enemy counteraction, we determine the average amount of weapons by the formula

$$n = \frac{\lg(1 - M_3)}{\lg(1 - MW_d)},$$

where W_d is the probability of the carrier reaching the target.

We will analyze an example by applying this method.

Example Eighteen: A strike is carried out by a rocket on a tank battalion in an area of concentration. The dimensions of the

area are 1 x 2 km. Determine the losses to personnel in the tanks (personnel made non-operative) if the radius of destruction of our weaponry is equal to 1.4 km.

The probable deviation of the rocket is 0.7 km.

Solution:

1. We determine the relative radius of destruction

$$R = \frac{1{,}4}{0{,}7} = 2.$$

2. We determine the relative dimensions of the target

$$Ц_x^{a} = \frac{1}{0{,}7} \approx 1{,}4; \quad Ц_y = \frac{2}{0{,}7} \approx 2{,}8.$$

Key: a - T.

3. From the graph, we find:

$$M_x = 0{,}81; \quad M_y = 0{,}76.$$

4. We determine the mathematical expectation of damage (the average percent of personnel of the tank battalion made non-functional):

$$M = 0{,}81 \cdot 0{,}76 \approx 0{,}62.$$

In this manner 62% of the personnel of the tank battalion are made non-functional, i. e. it is, for practical purposes, destroyed.

For the production of advance operative calculations of the needed weaponry, calculations are usually made in advance on the EVM tables, in which, for typical objectives and weapons the needed amount is given without considering enemy counter-action (the firing range amount).

Evaluating the Effectiveness of Weapons of Anti-Aircraft Devices

By effectiveness of anti-aircraft defense weapons, we shall mean their capability to fulfill a set of military objective of destroying aerospace devices of the enemy in flight.

The criterion for a quantitative analysis of military effectiveness stems from the specific nature of the task that a specific weapon is intended to perform.

In a number of cases, it is important to know not only the probability of destruction or the mathematical expectation of the number of destroyed targets, but ehe expected magnitude of one's own losses. In some cases, it is necessary to know the cost of one's expenditures (considering the expected losses) i. e., one must be able to make an evaluation of military economy.

Finally, such characteristics, as the time required to fulfill a military mission, have acquired great importance in modern circumstances.

However, as the most frequent criterion of anti-aircraft weapons, we use the probability of destruction of an enemy aerospace target or the mathematical expectation of the number of destroyed targets.

If several independent attacks are made on an enemy target, by identical weapons, (where each one attacks one time) then the probability of shooting down an aerospace target will be equal to

$$W_i = 1 - (1 - W)^i,$$

where W_i is the probability of shooting down a target in attacks of i weapons

W is the probability of shooting down a target with one weapon,

i is the number of independent attacking weapons.

When an aerospace target is attacked by different types of weapons (we will consider the attacks independent) the probability of shooting down the target will be equal to

$$\overset{a}{W_{\text{пор}}} = 1 - (1 - W_1)^{i_1}(1 - W_2)^{i_2} \ldots (1 - W_n)^{i_n},$$

Key: a - por

where $W_1, W_2, \ldots, W_n$ is the probability of shooting down a target with the 1, 2, . . . n-th type of weapon,

i_1 is the number of attacking weapons of the first type,

i_2 is the number of attacking weapons of the second type,

i_n is the number of attacking weapons of the n-th type.

Knowing the average probability of destruction of one aerial target, it is easy to determine the mathematical expectation of the number of destroyed enemy flying devices:

$$\overline{M} = \sum_{j=1}^{N_a} \overline{W}_j,$$

where M is the mathematical expectation of the number of destroyed targets,

N_t is the number of targets attacked by anti-aircraft weapons,

W_j is the probability of destroying one target by the j-th anti-aircraft weapon.

If the probability of destruction for all the weapons is the same, then the formula will take the form

$$M = N_{\text{ц}} W.$$

Key: a - T.

We will suppose, that N_t flying enemy devices participate in an enemy aerial attack (rockets, airplanes). To repel the attack, there are N anti-aircraft weapons (fighters, AA rockets, anti-rockets).

We will consider, that the anti-aircraft weapons are evenly distributed (in the attack) among the attacking group.

In this case, the probability of destroying one enemy flying apparatus will be

$$W^{a}_{\text{пор}} = 1 - (1 - W)^{\frac{N}{N_p}}.$$

Key: a - por;

The mathematical expectation of the number of destroyed targets is determined by the formula

$$M = N^{b}_{\text{ц}} W^{a}_{\text{пор}} = N^{b}_{\text{ц}} \left[1 - (1 - W^{\frac{N}{N_p}}\right].$$

Key: a - por; b - t.

We introduce the identity:

$$\frac{N}{N_p} = \lambda.$$

The quantity λ represents the correlation of forces in the air: the ratio of the number of attacking defense weapons to the number of flying enemy targets.

The connection between the probability of destruction and the quantity λ is established by the formula

$$W_{\text{пор}}^{a} = 1 - (1 - W)^{\lambda}.$$

Key: a - por.

The obtained formula allows one to solve the following important anti-aircraft problems:

-- to determine the relative losses of enemy flying devices from the given effectiveness W of the anti-aircraft weapon in one attack, and the correlation of forces in the air λ.

-- to determine the needed correlation of forces in the air from the known effectiveness W_{por} and the assigned magnitude of relative losses.

To solve problems of the second type, it is convenient to use the formula

$$\lambda = \frac{\lg (1 - W_{\text{пор}}^{a})}{\lg (1 - W)}.$$

Key: a - por.

In Table 5, values of the needed correlation of forces with respect to relative losses and weapon effectiveness are entered.

TABLE 5

	W	$W_{\text{пор}}^{b}$ 0,10	0,20	0,40	0,60	0,80	0,90	0,95
λ	0,20	3	1	1	1	1	1	1
	0,40	5	3	1	1	1	1	1
	0,60	9	5	3	1	1	1	1
	0,80	15	10	4	2	1	1	1
	0,90	22	13	6	4	2	1	1

Key: b - por

We will consider several examples of the application of the simplified method of evaluating the military possibilities of groups of anti-aircraft weapons.

Example Nineteen. In a massive enemy attack, we may expect up to 530 enemy airplanes.

The anti-aircraft forces have the following weapons:

-- 360 AA rocket launching pads

-- 350 fighters.

The probability of shooting down an enemy plane with a rocket is 0. 6; the probability of shooting down an enemy plane with a fighter if 0. 2 (calculating interception and the possibility of an attack in formation).

Considering that the enemy planes are attacked equally by AA rockets and fighters, determine the military capacity of the anti-aircraft system to repel the massive attack.

Solution:

1. We calculate the average number of attacks by AA rockets:

$$l_1 = \frac{360}{530} = 0{,}68.$$

2. We calculate the average number of fighter attacks

$$l_2 = \frac{350}{530} = 0{,}66.$$

3. We calculate the probability of destroying an enemy airplane with AA rockets:

$$W_1 = 1 - (1 - 0{,}6)^{0{,}68} = 0{,}46.$$

4. We calculate the probability of destroying an enemy airplane with fighters:

$$W_2 = 1 - (1 - 0{,}2)^{0{,}66} = 0{,}10.$$

5. We calculate the probability of shooting down an enemy airplane in a joint operation of AA rockets and fighters

$$W_{\text{пор}} = 1 - (1 - W_1)(1 - W_2) =$$
$$= 1 - (1 - 0{,}46)(1 - 0{,}10) = 1 - 0{,}54 \cdot 0{,}9 = 0{,}51.$$

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6. We calculate the mathematical expectation of the number of enemy airplanes shot down, by AA rockets and fighters:

$$M = N_{\text{ц}}^{a} W_{\text{пор}}^{b} = 530 \cdot 0{,}51 = 270.$$

Key: a- t; b - por.

Example Twenty: 400 rockets can participate in a simultaneous launch.

Spearheads of rockets are attacked by 300 anti-rockets. The probability of destroying a spearhead with one anti-rocket is equal to 0.4. Determine the military capabilities of the group of anti-rocket complexes.

Solution:

1. We calculate the average number of attacks, conducted against a spearhead:

$$l = \frac{300}{400} = 0{,}75.$$

2. We calculate the average probability of shooting down a spearhead.

$$W_{\text{пор}}^{b} = 1 - (1 - W)^{l} = 1 - (1 - 0{,}4)^{0{,}75} = 0{,}32.$$

Key: b - por.

3. We calculate the mathematical expectation of the number of destroyed spearheads:

$$M = N_{\text{ц}}^{a} W_{\text{пор}}^{b} = 300 \cdot 0{,}32 = 96.$$

Key: a - t; b - por.

Under present conditions, with the rapidly changing aero-space background, such a means of solving target-assignment is not sufficient.

The Commander will not have time to make a decision, supported by calculation and, in addition, the actual situation will be so complicated, and the number of possible variants so great, that quickly making the right decision, without special calculations, will be very difficult. From here, it follows that the problem of choosing a target-assignment, as the most important task in the dynamics of anti-aircraft defense operations, must be solved with the use of an electronic calculator.

Below, we have briefly laid out several means of solving target-assignment problems for different conditions, stemming from operational-tactical demands.

The problem of target-assignment for anti-aircraft defense weapons is very complicated in its full extent. In solving it, one must consider many factors: setting boundaries for the anti-aircraft defense weapon base with respect to the defended object or territory, the dimensions of zones in which enemy flying machines are observed by directional radio equipment, the temporary characteristics of channels, manueverability of targets, the possibility of applying active and passive obstacles, enemy counter-fire, and the presence of especially important targets in the composition of an attacking group.

All of these circumstances, to one degree or another, are considered in making concrete algorithms of target-assignment, which are used in machines designed for automatic control systems.

To clarify a principle aspect of the target-assignment problem, we will consider simplified schemes or models, in which target-assignment is determined from different points of view. This will help in clarifying basic principles, and the effects of target-assignment on anti-aircraft effectiveness, and to determine the degree of centralization necessary for anti-aircraft defense units, etc.

Example Twenty-One: From the conditions given in Example Nine determine the military capabilities of an anti-aircraft group, considering that fighters attack first, and then AA rockets attack those airplanes that have broken through.

Solution:

1. We calculate the average number of fighter attacks

$$i_1 = \frac{350}{530} = 0{,}66.$$

2. We calculate the probability of destroying an enemy plane with fighters

$$W_1 = 1 - (1 - 0{,}2)^{0{,}66} = 0{,}10.$$

3. We calculate the mathematical expectation of the number of enemy airplanes shot down by fighters

$$M_1 = 530 \cdot 0{,}10 = 53.$$

4. We calculate the mathematical expectation of the number of enemy airplanes breaking through the zone of fighter cover

$$M_{\text{пр}} \overset{a}{=} 530 - 53 = 477.$$

Key: a - pr.

5. We calculate the average number of AA rocket attacks

$$i_2 = \frac{360}{477} = 0{,}75.$$

6. We calculate the probability of destroying an airplane with AA rockets:

$$W_2 = 1 - (1 - 0{,}6)^{0{,}75} = 0{,}51.$$

7. We calculate the mathematical expectation of the number of enemy airplanes shot down by AA rockets:

$$M_2 = 477 \cdot 0{,}51 = 243.$$

8. We calculate the mathematical expectation of the number of enemy planes shot down by fighters and rockets (by their consecutive entry into the battle):

$$M = 53 + 243 = 296.$$

9. We determine the percentage of destroyed airplanes (the effectiveness of the anti-aircraft defense group):

$$W^{a}_{\text{уничт}} = \frac{296}{530} \cdot 100 = 56\%.$$

Key: a - dest.

Analyzing the probable variants of an enemy attack, and the diverse ways that its forces may be distributed, one may find optimal variants of repelling the attack, which must be worked out beforehand as characteristic.

Problems of this sort are most successfully solved by using mathematical models on electronic calculators.

Problems in Target Assignment for Anti-Aircraft Forces

Target-assignment is fixing a specific aero-space target for each type of anti-aircraft apparatus.

If there are several aero-space targets, whose attack we must repel, and we have at our disposal several weapons (fighters, AA rockets, anti-rocket rockets), then the organization of our fire-power begins with the making of a decision. In this decision, it is indicated which unit to aim at which target. It may turn out that several units will be aimed at the same target, but certain targets will not be fired upon.

Solving the problem of target-assignment is a typical example of a tactical decision, touching upon the means of applying already existing technology to battle.

Under the conditions of the last war, a decision of target-assignment was usually made by a Commander on the basis of experience and with regard to circumstances.

Target Assignment by Mathematical Expectation

The problem can be formulated in the following manner.

Suppose there are n anti-aircraft weapons, and there are N aerial targets. Each weapon makes on attack (launch or take-off), and may fire at any target, but without the same effectiveness. The probability of destroying each target, by any means, is given.

We must find an optimal target-assignment, i. e. one for which the mathematical expectation of the number of planes shot down is maximum.

As a result of the solution to this problem, each anti-aircraft weapon will be assigned a specific target, upon which it must fire, (here, it is possible, that one target will be fired at by several AA weapons).

We indicate the probability of destroying the j-th target with the i-th weapon by P_{ij}. The totality of the possibilities of destroying different targets by anti-aircraft weapons can be written in the form of a table, which is called a matrix of effectiveness.

$$\|P_{ij}\| = \begin{vmatrix} P_{11} & P_{12} & \dots & P_{1N} \\ P_{21} & P_{22} & \dots & P_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \dots & P_{nN} \end{vmatrix}.$$

In this matrix, for example, P_{21} is the probability of destroying target number one with the second anti-aircraft weapon, etc. The numbering of weapons and targets is done beforehand, but arbitrarily.

This problem is called a problem of target assignment of n X N.

Let us suppose that an anti-aircraft weapon is defending a certain territory. Then it is natural to assume, that the damage, which can be inflicted on the defended territory will be approximately proportionate to the number of targets (attacking), that break through. The task of the defense consists of keeping this damage to a minimum.

Therefore, as the criterion of effectiveness, we may choose the mathematical expectation of the number of targets shot down.

$$M = \sum_{j=1}^{N} W_j$$

where M is the mathematical expectation of the number of targets shot down,

W_j is the probability of shooting down the j-th target.

The probability of shooting down a target depends on the type of target-assignment, and in a given target-assignment is defined by the quantity P_{ij}.

It is necessary to find a target-assignment for which the mathematical expectation of the number of shot down targets will be maximum.

We will explain how the probability of shooting down a target is determined.

Suppose two weapons with numbers i and k fire at target number j.

The probability of not hitting the target with the i-th weapon is equal to $(1-P_{ij})$; the probability of not hitting the target with the k-th weapon is $(1-P_{jk})$.

The probability of not hitting the target when it is fired upon by the two weapons is

$$(1-P_{ij})(1-P_{kj}).$$

Therefore, the probability of hitting the target with the two weapons is

$$W_j = 1-(1-P_{ij})(1-P_{kj}).$$

Analogously, we may obtain the probability of shooting down a target with any number of anti-aircraft defense weapons.

Let us assume that the target-assignment has been completed, that is, that each anti-aircraft weapon is directed at a completely determined target.

This means that the dependency $j(i)$ is established; that the number of the target is determined synonymously by the number of the weapon (but not vice-versa, since in the case under consideration several weapons may fire at the same target.)

The function $j(i)$ is called the target-assignment function.

The problem of finding a target-assignment function can be solved by a direct analysis. Actually, the number of combinations for assigning n weapons among N targets if finite, but for large values of n and N, it is very great. If we analyze all the combinations, and for each of them calculate the mathematical expectation of the number of downed targets and find the target distribution $j(i)$, for which the mathematical expectation has the largest value, then the target-assignment problem will be solved. For large values of n and N a direct analysis of the possible variants is a long process, and now such problems are solved on electronic computers.

Examining the different variants for constructing a group of anti-aircraft weapons and the different variants of attack, one may obtain a recommendation for the most reasonable construction of the anti-aircraft group. The target-assignment problem allows clarification of the demands, presented to perspective anti-aircraft weapons when considering the question of repelling an attack by new or perspective weapons of aerial attack. Making a model of the problem on an electronic calculator, we may receive two important results, which will allow us to answer the following questions:

-- How to construct an anti-aircraft defense group, and what demands there are to be met with new weapons, so that the number of targets breaking through to the defended area is made minimum.

-- How to construct an attack by your own aviation, (order of battle, routes) so that the number of planes that break through is maximum.

The last problem is particularly important for the future of aviation.

In order to graphically portray the essence of the problem of target-assignment by mathematical expectation, we shall examine some examples.

Example Twenty-Two: A target-assignment problem of 2 X 2 is characterized by a matrix of probability of downing targets:

Weapon numbers	Target numbers 1	2
1	0,8	0,6
2	0,7	0,1

We must find the optimal target-assignment.

Solution:

We will write down all the variants of target-assignment as pairs of columns, where on the left we indicate the weapon number, and on the right the target number. In our case, four variants of target-assignment are possible.

$$\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ (1\ \ 1) & (1\ \ 1) & (1\ \ 2) & (1\ \ 2) \\ (2\ \ 1) & (2\ \ 2) & (2\ \ 2) & (2\ \ 1) \end{array}$$

The mathematical expectation of the number of shot down targets for each variant of target-assignment will be

$$M_{\text{I}} = 0{,}8 + (1 - 0{,}8) \cdot 0{,}7 = 0{,}94;$$
$$M_{\text{II}} = 0{,}8 + 0{,}1 = 0{,}9;$$
$$M_{\text{III}} = 0{,}6 + (1 - 0{,}6) \cdot 0{,}1 = 0{,}64;$$
$$M_{\text{IV}} = 0{,}6 + 0{,}7 = 1{,}3.$$

Therefore, the most effective is variant IV. In this variant, the first weapon does not fire at the first target, in spite of the fact that it has the most effectiveness when firing at it.

Example Twenty-Three: The target assignment problem is given by the matrix:

Weapon Numbers	Target numbers 1	2
1	0,6	0,1
2	0,8	0,2

We must find the optimal target assignment.

Solution: The possible target-assignment variants are as follows:

$$\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix} & \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} & \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix} & \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} \end{array}$$

The mathematical expectation of the number of targets shot down for each variant will be :

$$M_{\text{I}} = 0{,}6 + (1 - 0{,}6)\cdot 0{,}8 = 0{,}92;$$
$$M_{\text{II}} = 0{,}6 + 0{,}2 = 0{,}8;$$
$$M_{\text{III}} = 0{,}1 + (1 - 0{,}1)\cdot 0{,}2 = 0{,}28;$$
$$M_{\text{IV}} = 0{,}1 + 0{,}8 = 0{,}9.$$

In the given case, variant I is the optimal, i. e., both weapons must fire at the first target.

The Target Assignment Problem with Consideration of Importance of Targets.

In repelling an enemy aerial attack, that is especially massive, it is necessary to consider the danger which one or another target presents. In such an approach to the organization of reflecting an enemy attack, it is necessary to add coefficients of value to the targets, which are also known as the "weight" of the targets.

Increased "weights" may be assigned to the carriers of nuclear weapons (if they are distinguished by some sign) to reconnaisance

devices, hindrance producers, and targets located on the threshold of destruction.

Suppose separate targets are assigned weights K_1, K_2, ... K_n, expressing the numbered degree of danger from the targets. As a criterion of effectiveness in such a situation, we must apply not only the mathematical expectation of the number of targets shot down, but the so-called "weighted" number i. e. including the importance of the target:

$$M = \sum_{j=1}^{N} K_j W_j.$$

Methods of finding the optimal target-assignment remain the same, but with the difference, that in the matrix of probabilities of shot down targets each column is multiplied by the "weight" (level of importance) of the corresponding target.

Target-Assignment by Mathematical Expectation, But When Firing at the Maximum Possible Number of Targets

Target-assignment by the mathematical expectation of the number of shot down targets sometimes leads to several targets not being fired at. This occurs, when the probability of hitting certain targets is so small, that it is more useful not to use a weapon by itself, but with the help of other ones. Target-assignment, based on such a principle, can lead to the enemy, having understood our tactics, deliberately drawing fire away from important targets to less important ones, placing the latter into situations where the effectiveness of firing at them will be high.

Therefore, it is necessary also to have other principles of target-assignment, which contain a limit to the number of targets which pass through without being fired upon. It is efficient to make the following demand: to fire at every target, if it is possible, and to concentrate the fire of several weapons on one target only when all possible targets have fired upon one time.

With this limitation, the optimal target assignment also corresponds with the maximum mathematical expectation of the number of shot down targets.

In such a target-problem situation, there can only be targets not fired upon when the number of anti-aircraft defense weapons is less than the number of targets. In order to prevent other weapons from firing at a target that has already been fired upon, one applies, in practice, the so-called "prohibition markers". These are special signs or symbols, which are supplied to targets that have been fired upon on the aerial situation screens or in the memory banks of the electronic computers, that are automatically controlling the weapons.

We will examine a target-assignment problem of n X N, where n is less than N.

With this condition, each weapon must be directed at only one target, and each target must be fired at by only one weapon. The target-assignment j(i) defines the mutually identical correspondence between the numbers of the weapons and of the targets.

The number of possible target-assignment variants will, in the given case, be equal to

$$N(N-1)(N-2)\ldots(N-n+1).$$

Example Twenty-Four: We have a target-assignment of 2 X 3, i. e. two anti-aircraft weapons for three targets. The matrix of shoot-down probability takes the form:

Number of Weapons	Number of Targets		
	1	2	3
1	0,6	0,4	0,3
2	0,8	0,2	0,4

Find the optimal target-assignment.

Solution:

In examining the example, we see how many variants of target-assignment there are in all: 3 X 2 = 6.

These variants are as follows:

I	II	III	IV	V	VI
(1 1)	(1 1)	(1 2)	(1 2)	(1 3)	(1 3)
(2 2)	(2 3)	(2 1)	(2 3)	(2 2)	(2 1)

The mathematical expectation of the number of shot down targets for each variant of target-assignment will be:

$$M_{I} = 0{,}6 + 0{,}2 = 0{,}8;$$
$$M_{II} = 0{,}6 + 0{,}4 = 1{,}0;$$
$$M_{III} = 0{,}4 + 0{,}8 = 1{,}2;$$
$$M_{IV} = 0{,}4 + 0{,}4 = 0{,}8;$$
$$M_{V} = 0{,}3 + 0{,}2 = 0{,}5;$$
$$M_{VI} = 0{,}3 + 0{,}8 = 1{,}1.$$

The optimal target-assignment corresponds to variant III; the first weapon fires at the second target, and the second weapons fires at the first target.

Target Assignment by Probability

In modern conditions, the important role of anti-aircraft is to destroy all targets. This might be in defending an important objective, when the enemy is using nuclear weapons, and the penetration of even one of the targets is enough that the objective will be destroyed. In such cases, the effectiveness of the anti-aircraft defense cannot be judged by the mathematical expectation of the number of targets shot down, since one cannot allow the penetration of even one carrier to the objective.

Therefore, it is necessary to take, as the criterion of effectiveness, the probability of not allowing a single target to get to its objective. Suppose we are given a matrix of probabilities of shot down targets by anti-aircraft weapons as follows:

$$\|P_{ij}\| = \begin{vmatrix} P_{11} & P_{12} & \dots & P_{1N} \\ P_{21} & P_{22} & \dots & P_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \dots & P_{nN} \end{vmatrix}.$$

We must find the target-assignment, for which the probability of destroying all of the targets has the maximum value.

Obviously, for this all of the targets must be fired at and the problem has a solution only if i is greater than or equal to j, (we are examining a case, where each weapon fires only once).

If the probability of shooting down the j-th target by firing at it with several weapons is equal to W_j, then the probability of destroying all the targets (considering that the events, that compose the destruction of separate targets, to be independent) is equal to

$$W = W_1 W_2 \dots W_N.$$

The optimum target-assignment will correspond to the maximum value of the probability of destroying all of the targets.

<u>Example Twenty-Five:</u> We are given a matrix of shoot-down probabilities of 2 X 2:

Number of Weapons	Number of Targets	
	1	2
1	0,6	0,9
2	0,8	0,7

Determine the optimum target-assignment by probability.

We examine the possible variants of target-assignment:

I	II	III	IV
(1 1)	(1 1)	(1 2)	(1 2)
(2 1)	(2 2)	(2 2)	(2 1)

Of these four variants, two (I and III), are clearly useless for solving the problem of target-assignment by probability, since, in them, only one target is fired at.

The probability of shooting down both targets in variants II and IV will be equal to

$$W_{II} = 0{,}6 \cdot 0{,}7 = 0{,}42;$$
$$W_{IV} = 0{,}8 \cdot 0{,}9 = 0{,}72.$$

The optimum target-assignment by probability corresponds to variant IV: the first weapon fires at the second target, and the second weapon at the first target.

Target-Assignment for Identical Shoot-down Probabilities

In practice, one often meets the case, where anti-aircraft weapons have approximately the same probability of shooting down enemy targets. This, for example, corresponds to the case of repelling an attack by targets of one type, by using defense fighters with the same tactical and flight specifications, or anti-aircraft rocket complexes with the same characteristics.

We will consider a target-assignment problem of $n \times N$ with identical shoot-down probabilities.

$$P_{11} = P_{12} = \ldots = P_{nN} = W.$$

This problem can be solved with two criteria of effectiveness:

-- by mathematical expectation;

-- by shoot-down probability,

We solve the problem by mathematical expectation.

If n is less than N, the solution is trivial: each weapon may fire at any target, but no two weapons may fire at the same target.

Now, suppose that n is greater than N. We will consider that the number of weapons is a multiple of the number of targets:

$$n = KN,$$

where K is a whole number.

It can be proven, that the optimum target-assignment can be reached, if we distribute the weapons equally among the targets, with K weapons for each target.

Then the probability of shooting each target will be:

$$W_j = 1 - (1 - W)^K.$$

We will consider a more general case, when the number of weapons is not a multiple of the number of targets.

$$n = KN + l.$$

In this case, it is useful to distribute the anti-aircraft weapons in the following manner: at each of ℓ targets we direct K + 1 weapons, and K weapons at each of N - ℓ targets.

Therefore, the optimum target-assignment for identical shoot-down probabilities corresponds to the most even distribution of anti-aircraft weapons among the targets.

We analyze the case, when as a criterion of effectiveness, we use the probability that not one target will get past the line of defense.

The probability of shooting one target will be equal to

$$W_j = 1 - (1 - W)^K,$$

where W_j is the probability of shooting down the j-th target,

W is the probability of shooting down a target with one anti-aircraft weapon,

K is the number of anti-aircraft weapons, firing at the target.

The probability of shooting down all the targets will be

$$W^{a}_{\text{п}} = [1 - (1 - W)^K]^N.$$

Key: a - por

If we change one of the weapons from one target to another, the criterion of effectiveness will become equal to

$$W'^{a}_{\text{пор}} = [1-(1-W)^{K}]^{N-2}[1-(1-W)^{K+1}] \times \times [1-(1-W)^{K-1}].$$

Key: a - por

Calculating W'por from Wpor, we obtain

$$W_{\text{пор}}^{a} - W'^{a}_{\text{пор}} = [1-(1-W)^{K}]^{N-2}(1-W)^{K-1}W^{2} > 0.$$

Key: a - por

Here it follows that W_{por} if greater than W'_{por}, i.e. the optimum target-assignment also corresponds to the even distribution of weapons on targets.

Combat Effectiveness of Fighter Aviation

The probability of shooting down an aerial target can serve as the criterion for evaluating the combat effectiveness of fighter aviation.

The probability of destruction, as the probability of the combination of several events, will be equal to

$$W^{a}_{\text{пор}} = W^{b}_{\text{пер}} W^{c}_{\text{обн}} W^{d}_{\text{ат}} W^{e}_{\text{сб}},$$

Key: a - por; b - int; c - obs; d - at; e - sd.

where W_{por} is the probability of destroying an aerial target,

W_{int} is the probability of interception,

W_{obs} is the probability of observing the aerial target with fighter equipment,

W_{at} is the probability of the attack (the probability that the fighter will be located in an area, where it can perform a maneuver to occupy a starting position for the attack).

W_{sd} is the shoot-down probability.

If an aerial target is attacked by several different types of fighters, then the probability of destruction will be equal to

$$W_{nop} = 1 - (1 - W_1)^{i_1} (1 - W_2)^{i_2} \ldots (1 - W_n)^{i_n},$$

Key: a - por

where W_{por} is the probability of destroying an aerial target

W_1, W_2, ... W_n are the probabilities of destroying a target in one attack, for different fighters,

i_1, i_2, ..., i_n are the number of attacks on the aerial target by different fighters.

It should be borne in mind that, depending on the correlation of forces (attacking enemy targets and fighters sent to repel them), the quantities i_1, i_2, i_n can be less than one.

To compute the probability of destroying an aerial target in several attacks by fighters, we may use Table 6.

TABLE 6

Average number of attacks per target, i	Probability of destruction in one attack W_{por}				
	0,25	0,50	0,75	0,80	0,90
0,2	0,06	0,13	0,24	0,37	0,50
0,4	0,11	0,24	0,42	0,48	0,60
0,6	0,16	0,34	0,56	0,62	0,75
0,8	0,21	0,42	0,67	0,72	0,84
1,0	0,25	0,50	0,75	0 80	0,90
2,0	0,44	0,75	0,94	0,96	0,99
3,0	0,58	0,88	0,98	0,99	0,999

To calculate the mathematical expectation of the number of destroyed enemy targets, one may use the formula (for identical probabilities of destruction):

$$M = N_{\text{ц}} W_{\text{пор}},$$

Key: a - t ; b - por

where M is the mathematical expectation of the number of destroyed enemy targets,

N_t is the number of attacking targets,

W_{por} is the probability of destroying an aerial target (taken to be the same for all targets).

We will consider several examples of calculating the probability of destruction.

Example Twenty-Six: Determine the probability of destroying a bomber with fighters armed with cannons for W = 0.25 (the probability of a simultaneous occurrence of events: observation, attack and shooting down with answering fire included in calculations). The number of attacks is 2.

Solution:

$$W_{\text{ИА}}^{a} = 1 - (1 - 0{,}25)^2 = 0{,}44.$$

Key: a - IA.

If 50 bombers participated in an attack and they were each attacked twice, then the mathematical expectation of the number of bombers shot down would be

$$M = 50 \cdot 0{,}44 = 22.$$

Example Twenty-Seven: Determine the probability of destruction and the mathematical expectation of the number of bombers shot down, if each bomber is attacked once by a fighter with cannon armament (W_1 = 0.25) and an anti-aircraft guided shell is fired at every other bomber (W_2 = 0.75) without considering interference or answering fire from the bomber.

Solution:

$$W_{\underset{a}{ИА}} = 1-(1-0{,}25)\,(1-0{,}75)^{0{,}5} = \\ = 1-0{,}75\cdot 0{,}5 = 0{,}62. \\ M = 50\cdot 0{,}62 = 31.$$

Key: a - IA.

Example Twenty-Eight: Determine the probability of destruction and the mathematical expectation of the number of bombers shot down (100 bombers are participating in the attack).

In the first variant, all of the bombers are attacked evenly by 100 fighters with cannon armament ($W_1 = 0.25$), 20 fighters with winged air-to-air rockets ($W_2 = 0.9$), and 20 AA (ground-to-air) rockets ($W_3 = 0.75$).

In the second variant the bomber attack is in four groups with 25 airplanes in each group.

The distribution of forces by group is entered in Table 7.

TABLE 7

a Группа	b Средства		
	c истребительная авиация с пушками	d истребительная авиация с УРС	e ЗУР
I	50	10	10
II	25	5	5
III	13	5	5
IV	12	0	0

Key: a - group; b - weapons; c - fighter aviation with cannons; d - fighter aviation with guided rockets; e - AA rockets.

Solution:

We determine the probability of destruction and the mathematical expectation of the number of bombers shot down in the first attack variant.

$$W^{a}_{HA} = 1 - (1 - 0{,}25)\,(1 - 0{,}9)^{0{,}2}\,(1 - 0{,}75)^{0{,}2} =$$
$$= 1 - 0{,}75 \cdot 0{,}50 \cdot 0{,}76 = 0{,}72;$$
$$M = 100 \cdot 0{,}72 = 72.$$

Key: a - IA

For the second variant, we will determine the probability of destruction and the mathematical expectation separately for each group.

$$\underset{a}{W_{HAI}} = (1 - 0{,}25)^2\,(1 - 0{,}75)^{0{,}4}\,(1 - 0{,}9)^{0{,}4} =$$
$$= 1 - 0{,}56 \cdot 0{,}58 \cdot 0{,}40 = 0{,}88;$$
$$M_I = 25 \cdot 0{,}88 = 22;$$
$$\underset{a}{W_{HAII}} = 1 - (1 - 0{,}25)\,(1 - 0{,}75)^{0{,}2}\,(1 - 0{,}9)^{0{,}2} =$$
$$= 1 - 0{,}75 \cdot 0{,}76 \cdot 0{,}50 = 0{,}72;$$
$$M_{II} = 25 \cdot 0{,}72 = 18;$$
$$\underset{a}{W_{HAIII}} = 1 - (1 - 0{,}25)^{0{,}6}\,(1 - 0{,}75)^{0{,}2}\,(1 - 0{,}9)^{0{,}2} =$$
$$= 1 - 0{,}84 \cdot 0{,}76 \cdot 0{,}50 = 0{,}67;$$
$$M_{III} = 25 \cdot 0{,}67 \approx 17;$$
$$W_{HAIV} = 1 - (1 - 0{,}25)^{0{,}5} = 0{,}13;$$
$$M_{IV} = 25 \cdot 0{,}13 \approx 3.$$

Key: a - IA

The general number of bombers shot down in the second variant of attack and the given distribution of forces is

$$M = 22 + 18 + 17 + 3 = 60.$$

From the analyzed example, it is apparent that the tactic of evenly distributing weapons among the bombers is useful for anti-aircraft defense. Therefore, it is more convenient for the bombers to arrange their attack, so as to cause a more uneven distribution of fighter forces among the attacking group.

The Military Effectiveness of Bomber Aviation.

As the criterion of evaluating the military effectiveness of bombers, we may use such quantities, as the probability of destroying

certain separate targets, or the mathematical expectation of the area of the target destroyed, (the mathematical expectation of damage inflicted on the target).

In addition, it is still necessary to consider the technical reliability of the airplane (or winged rocket).

A combat flight can be conditionally devided into the following stages:

-- the flight to the target, where the airplane is subjected to the influence of different flying devices of the fighter type.

-- entry into the target area with the use of corresponding navigational devices,

-- using weapons against the target,

-- return to the airport (for apparati of repeated use).

All targets are divided into two groups. To the first group belong small targets, for whose destruction one hit by the correctly chosen device will be sufficient. To the second group belong area targets, for whose destruction several hits are necessary.

As the criterion of combat effectiveness for destroying small targets, we can use the probability of destroying a target.

Considering the probability of each stage (event), we present the criterion of effectiveness in the form of the probability of destroying a separate target:

$$W_{\text{пор}} = W_{\text{д}} W_{\text{т}} W_{\text{б}},$$

Key: a - por; b - d; c - t; d - b.

Where W_{por} is the probability of destroying a separate target,

W_d is the probability of technical reliability of the flying device,

W_b is the probability of destroying the target, under the condition that the bombers reach it.

The probability of reaching the target is the probability of the opposite event with respect to the bombers being destroyed by fighters. It can be written

$$W_x = (1 - W_1)^{i_1}(1 - W_2)^{i_2} \ldots (1 - W_n)^{i_n}$$

Key: a-d.

or for small W_1, W_2, ... W_n

$$W_x = e^{-(i_1 W_1 + i_2 W_2 + \ldots + i_n W_n)},$$

where W_1, W_2, ..., W_n are the probabilities of shooting down bombers with fighter type equipment,

i_1, i_2, ... i_n are the number of attacks by different types of fighter equipment.

We will consider now the probability of destroying a target, which can be knocked out of commission by one hit. We will indicate the probability of a hit with one bomb by the letter P_6. Then the probability of destroying the target with even one bomb will be

$$W_6 = 1 - (1 - P_6)^{n_6},$$

where P_6 is the probability of hitting the target with one bomb,

n_6 is the number of bombs dropped as a volley.

The probability of a hit when one bomb is dropped, (the probability of hitting a rectangle) is equal to

$$P_6 = \hat{\Phi}\left(\frac{a}{E}\right)\hat{\Phi}\left(\frac{b}{E}\right),$$

Where $\hat{\phi}$ is Laplace's Function

a, b are the dimensions of the target

E is the probable deviation of a bomb.

Finally, the probability of fulfilling the mission, the destruction of a separate small target, can be written in the form

$$W_{пор} = W_т [1 - (1 - P_б)^{n_б}] \, e^{-(l_1 W_1 + l_2 W_2 + \dots + l_n W_n)}.$$

Key: a - por.

We will consider bomber operations on area targets.

Suppose objectives are laid out on a certain area, and it is not expedient to distinguish them as independent targets. In this case, combat effectiveness can be judged by the magnitude of the mathematical expectation of the area that falls into the zone of destruction (to a given degree).

If, without considering counterfire and technical reliability, the mathematical expectation of the relative damage for one bomber is equal to M, then the mathematical expectation of relative damage, considering counterfire and technical reliability, will be equal to

$$M_{пр} = M W_т W_д,$$

Key: a - pr; b - t; c - d.

where M_{pr} is the mathematical expectation of the relative damage considering enemy counterfire and technical reliability,

W_t is the probability of technical reliability,

W_d is the probability of reaching the target.

The probability of reaching the target is determined by the formulas introduced before. The probability of technical reliability must be given on the basis of experience.

In this way, determining a criterion of military effectiveness of flying devices of the bomber type for operations against area targets merges at the base with determining the unconditional mathematical expectation (without considering enemy fire and technical reliability) of the relative damage inflicted on a target by one flying device.

If n independent launchings (bombings) are conducted on a target, and in each separate case the relative damage is the same and equal to M, then we can write

$$M_n = 1 - (1 - M)^n,$$

where M_n is the mathematical expectation of the relative damage inflicted on the target in n independent launchings (bombings),

M is the mathematical expectation of relative damage, inflicted on the target in one launch (bombing), and

n is the number of firings (bombings).

Most often, it is necessary to know how many units must be dropped on the target, in order to inflict a given amount of damage.

From the preceding formula we may obtain the following expression for n, without considering enemy counterfire or technical reliability (for independent bombings):

$$n = \frac{\lg(1 - M_n)}{\lg(1 - M)}.$$

This quantity is called the firing range array of weapons.

In Tablc 6, we have entered values of M, depending on the mathematical expectation of the relative damage in one bombing M, and on the number of bombings n.

TABLE 8

n	M								
	0,10	0,20	0,30	0,40	0,50	0,60	0,70	0,80	0,90
1	0,10	0,20	0,30	0,40	0,50	0,60	0,70	0,80	0,90
2	0,19	0,36	0,51	0,64	0,75	0,84	0,91	0,96	0,99
3	0,27	0,46	0,66	0,78	0,88	0,94	0,97	0,90	1,00
4	0,34	0,59	0,76	0,87	0,94	0,97	0,99	1,00	—
5	0,41	0,67	0,83	0,92	0,97	0,99	1,00	—	—

From the table, it follows that, for instance, to inflict damage over 50% of the target area, it is necessary to have the following number of bombings:

-- one, if M = 0.5,

-- two, if M = 0. 3,

-- three, if M = 0. 2.

Example Twenty Nine: Compare the combat effectiveness of piloted bombers and winged and ballistic rockets in an operation against the runway of an airfield with a metal cover 2500 X 100 m.

The relative radii of destruction (expressed in probable deviations) are:

-- 3 for piloted bombers,

-- 6 for winged rockets,

-- 0. 6 for ballistic rockets.

The probability of reaching the target, for a ballistic rocket, is equal to one. For a winged rocket, it is 0. 8. The technical realiability probability for a ballistic rocket is 0. 9; for a bomber or winged rocket it is 1. 0.

On the average, piloted bombers are attacked once by fighters with cannon armament (W_{IA} = 0. 25), and every third bomber is attacked by an AA rocket (W_{ar} = 0. 7).

Solution:

1. Relative radii of destruction:

$$R_p = 3; \quad R_{wr} = 6; \quad R_{br} = 0.6.$$

2. We determine the relative dimensions of the target (expressed in units of probable deviation). Let:

$$Ц_{x\,\text{сам}}^{a} = 12{,}5; \quad Ц_{y\,\text{сам}}^{a} = 0{,}5; \quad Ц_{x\,\text{к.р}}^{b} = 25;$$
$$Ц_{y\,\text{к.р}}^{b} = 1; \quad Ц_{x\,\text{б.р}}^{c} = 2{,}5; \quad Ц_{y\,\text{б.р}}^{c} = 0{,}1.$$

Key: a - p; b - wr; c - br; d-T.

3. By the graph (Fig. 13) we determine the mathematical expectation of the length of the area of overlap for corresponding values of R

and T. Then, multiplying them, we obtain the quantities M:

$$M_{\text{сам}}^{a} = 0{,}5 \cdot 0{,}95 = 0{,}48; \quad M_{\text{к.р}}^{b} = 0{,}50 \cdot 1 = 0{,}50;$$
$$M_{\text{б.р}}^{c} = 0{,}29 \cdot 0{,}28 = 0{,}08.$$

Key: a - p; b - wr; c - br.

4. We determine the probability of reaching the target for the bombers:

$$W_{\text{д}}^{c} = (1 - W_{\text{ИА}}^{c})^{i_{\text{ИА}}} (1 - W_{\text{з.р}}^{b})^{i_{\text{з.р}}^{b}} =$$
$$= (1 - 0{,}25)(1 - 0{,}7)^{0{,}33} = 0{,}52.$$

Key: a - d; b - ar; c - IA.

5. We compute the criterion of combat effectiveness:

$$M_{\text{сам.пр}}^{a} = 0{,}48 \cdot 1 \cdot 0{,}52 = 0{,}25;$$
$$M_{\text{к.р.пр}}^{b} = 0{,}5 \cdot 1 \cdot 0{,}80 = 0{,}40;$$
$$M_{\text{б.р.пр}}^{c} = 0{,}08 \cdot 0{,}9 \cdot 1 = 0{,}07.$$

Key: a - p. pr; b - wr. pr; c - br. pr.

In this way, for the given conditions, winged rockets have the most combat effectiveness, and ballistic rockets - the least.

Example Thirty: A strike is made against an objective whose dimensions are 4 X 4 km by piloted bombers, winged, and ballistic rockets. Compare the effectiveness if the relative radii of destruction are $R_p = 20$, $R_{wr} = 4$, and $R_{br} = 2.7$.

The objective is protected by anti-aircraft rockets with such a density, that on the average each bomber and each winged rocket is attacked one time. The probability of being shot down in one attack is: for a bomber, 0.7; for a winged rocket, 0.5. The probability of a ballistic rocket reaching the target is 0.9. The technical reliability probability of all these weapons is one.

Solution:

1. We determine the relative dimensions of the target:

$$\overset{a}{Ц}{}_{x\,\text{сам}}^{\,b} = Ц_{y\,\text{сам}} = 10;\quad Ц_{x\,\text{к.р}}^{\,c} = Ц_{y\,\text{к.р}} = 2;$$
$$Ц_{x\,\text{б.р}}^{\,d} = Ц_{y\,\text{б.р}} = 1{,}3.$$

Key: a - T; b - p; c - wr; d - br.

2. $$M_{\text{сам}}^{a} = 1\cdot 1 = 1;\quad M_{\text{к.р}}^{b} = 0{,}98\cdot 0{,}98 = 0{,}96;$$
$$M_{\text{б.р}}^{c} = 0{,}92\cdot 0{,}92 = 0{,}85.$$

Key: a - p; b - wr; c - br.

3. $$W_{\text{д.сам}}^{a} = 1 - 0{,}7 = 0{,}3;\quad W_{\text{д к.р}}^{b} = 1 - 0{,}5 = 0{,}5.$$

Key: a - d. p, b - d. wr.

4. $$M_{\text{сам.пр}}^{a} = 1\cdot 1\cdot 0{,}3 = 0{,}3;\quad M_{\text{к.р.пр}}^{b} = 0{,}96\cdot 1\cdot 0{,}5 = 0{,}48;$$
$$M_{\text{б.р.пр}}^{c} = 0{,}85\cdot 1\cdot 0{,}9 = 0{,}77.$$

Key: a - p. pr; b - wr. pr; c - br. pr.

From the example, it is apparent, that in the solution of this problem, ballistic rockets have the most combat effectiveness, and piloted bombers, the least.

Example Thirty-One: A strike is made on a target objective 20 X 10 km.

The relative radii of destruction are equal to

$$R_{\text{сам}} = 35;\quad R_{\text{к.р}} = 3{,}5;\quad R_{\text{б.р}} = 2{,}3.$$

Key: a- p; b - wr; c - br.

Anti-aircraft defense along the route is such that, on the average, each bomber is attacked twice by fighters with cannon, ($W_{I\Lambda} = 0.25$) and an anti-aircraft rocket is fired at every fourth bomber ($W_{ar} = 0.7$). Every second winged rocket, in approaching the target area, is attacked once by an AA rocket ($W_{ar} = 0.5$). The technical reliability probability of the winged and ballistic rockets is 0. 9.

Compare the combat effectiveness of the flying devices.

Solution:

1. The relative radii of destruction:

$$R^{a}_{\text{сэм}} = 35; \quad R^{b}_{\text{к.р}} = 3{,}5; \quad R^{c}_{\text{б.р}} = 2{,}3.$$

Key: a - p; b - wr; c - br.

2. Determine the relative dimensions of the target:

$$Ц_{x\,\text{сэм}}^{a} = 50; \quad Ц_{y\,\text{сэм}} = 25;$$

$$Ц_{x\,\text{к.р}} = 5; \quad Ц_{y\,\text{к.р}} = 2{,}5;$$

$$Ц_{x\,\text{б.р}} = 3{,}3; \quad Ц_{y\,\text{б.р}} = 1{,}7.$$

Key: a - p

3. Determine the mathematical expectation of damage:

$$M_{\text{сэм}} = 1 \cdot 1 = 1; \quad M_{\text{к.р}} = 0{,}91 \cdot 0{,}96 = 0{,}87;$$

$$M_{\text{б.р}} = 0{,}86 \cdot 0{,}90 = 0{,}77.$$

Key: a - p.

4. Determine the probability of reaching the target:

$$W_{\text{д.сэм}} = (1 - 0{,}25)^2 (1 - 0{,}7)^{0{,}25} = 0{,}56 \cdot 0{,}74 = 0{,}42;$$

$$W_{\text{д.к.р}} = (1 - 0{,}5)^{0{,}5} = 0{,}7.$$

Key: a - d. p; b - d. wr.

5. Compute the criterion of military effectiveness:

$$M^{a}_{\text{сэм.пр}} = 1 \cdot 1 \cdot 0{,}42 = 0{,}42;$$

$$M^{b}_{\text{к.р.пр}} = 0{,}87 \cdot 0{,}9 \cdot 0{,}7 = 0{,}55;$$

$$M^{c}_{\text{б.р.пр}} = 0{,}77 \cdot 0{,}9 \cdot 1 = 0{,}69.$$

Key: a - p. pr; b - wr. pr; c - br. pr.

In this example, ballistic rockets also have the greatest combat effectiveness, and piloted bombers, the least.

Calculating the Array of Air Weapons Needed to Decide a Combat Mission

All problems that are involved with evaluating the effectiveness of fire on single, group, and area targets are related to the type known as direct problems.

A direct problem is set in the following manner. A certain detachment of weapons is assigned to solve a given military problem. One must determine a criterion of effectiveness.

For practical planning of air operations, we are interested not only in direct, but in inverse problems. In this case, we are given a value for the criterion of effectiveness; we must determine the combat conditions, for which the criterion of effectiveness reaches this value. Inverse problems are of great significance in practical planning of combat air operations. To this group belong problems involved in calculating the array of weapons, needed to solve a military mission.

One may introduce the following examples of calculating an array of air weapons:

-- how many bombers are needed (fighter-bombers, winged rockets) to destroy a single target with a given probability,

-- how many weapons to apply against a group target, so that the average number of destroyed units reaches a given value,

-- how many fighters must be dispatched so that in repelling an attack, not less than K enemy units will be destroyed with a given probability,

-- how many bombings are needed against an area target, so that the average relative destruction of the area of the target and the probability of destroying not less than K% of the area of the target reach their assigned values.

All problems of calculating the array of weapons are solved by the same methods. The scheme for solving a problem in determining an array of weapons can be given in the following sequence.

-- we choose a criterion of effectiveness (this can be the probability of fulfilling a mission W_3)

-- we calculate the probability of destroying the target with one "shot" (bombing, launching), W,

-- we calculate the probability of destroying the target in several (independent) shots:

$$W_n = 1 - (1 - W)^n;$$

-- we set the probability of destroying the target W_3,

-- we determine the needed number of "shots" to fulfill the combat mission

$$n = \frac{\lg(1 - W_3)}{\lg(1 - W)}.$$

Example Thirty Two: The probability of destroying an enemy ship with one bomber with the usual means is equal to 0.2. How many bombers are needed to destroy a ship with a probability of 0.7?

Solution:

In our example $W_3 = 0.8$, $W = 0.2$.

Placing these figures into the formula, we obtain the needed number of bombers.

$$n = \frac{\lg(1 - 0{,}7)}{\lg(1 - 0{,}2)} = 5.$$

If the number of weapons needed to destroy an area target is being determined, we must use the formula

$$n = \frac{\lg(1 - M_3)}{\lg(1 - M)}.$$

Example Thirty-Three: One bomber inflicts on a town relative damage of $M = 0.2$ (destroys 20% of the area of the target). How many bombings are necessary (bombers reaching the target) to invlict damage of $M_3 = 0.8$?

Solution:

Substituting the given quantities in the formula, we have

$$n = \frac{\lg(1-0{,}8)}{\lg(1-0{,}2)} = \frac{-0{,}699}{-0{,}096} = 7.$$

The same results may be obtained with the aid of a graph (Fig. 12).

The Target-Assignment Problem for Bomber Aviation

Solving a target-assignment problem is the basis for planning the application of bomber aviation.

As a basis for solving a target-assignment problem, we may use an evaluation of the effectiveness of bombing separate targets which make up a complex.

If we have the problem of choosing a target-assignment on the condition of obtaining the maximum average total relative damage, or on the condition that the average damage must have a given value, then we must speak about target-assignment by mathematical expactation.

If we are given a target-assignment problem on the condition of reaching a maximum or given value of probability of fulfilling a combat mission, then we can speak of target-assignment by probability of fulfilling a combat mission.

The Problem of Target Assignment by the Mathematical Expectation of Relative Damage.

This problem is solved in this sequence

Suppose we are given enemy targets and different aviation weapons.

First, we will determine the average amounts of damage, inflicted on the targets by the explosion of one, two, or more devices. Then we will calculate the amounts of relative damage as the ratio of the average amounts of damage to the total area of all the targets. We will set up a matrix, whose elements are amounts of relative damage.

Now, the optimum target-assignment problem becomes a problem of distribution of (explosive) devices among targets, to yield a maximum mathematical expectation of total relative damage.

We will consider the order of solving a target-assignment with an example.

Example Thirty-Four: There are two equally vulnerable targets, whose dimensions in units of probable deviation, are equal. For the first target $T_x = 2$, $T_y = 3$. For the second target $T_x = 4$, $T_y = 3$. We may apply devices against the targets which, upon exploding, produce an area of destruction with a radius, expressed in units of probable deviation of $R = 2$. There are four such devices. Determine the optimum target-assignment, using the maximum mathematical expectation of the total relative damage.

Solution:

First, we make calculations to evaluate the effectiveness of bombing each target. From the graph (Fig. 13) we determine the mathematical expectation M of relative damage. In our example:

-- for the first target $M_1 = 0.214$,

-- for the second target $M_2 = 0.180$.

The average relative area of destruction (damage), accumulated on one target in several bombings, is determined by the formula

$$M_n = 1 - (1 - M)^n.$$

Completing the calculations, we obtain:

-- for the first target -
for two bombings - 0.387,
for three bombings - 0.512,

for four bombings - 0.615,

-- for the second target:

for two bombings - 0.335,
for three bombings - 0.445,
for four bombings - 0.540.

We set up a matrix of amounts of relative damage.

Number of Bombings	Target Number 1	Target Number 2
1	0,124	0,180
2	0,387	0,335
3	0,512	0,445
4	0,615	0,540

In the first column of the matrix are placed amounts of relative damage, taken in relation to the area of the first target. In the second column, they are taken in relationship to the area of the second target. Such amounts of relative damage cannot be compared with each other. One must go to general units of damage. For this, we may recalculate the amounts of relative damage by absolute units, or by relative units determined by their relationship to the sum area of both targets, i. e. by dividing absolute amounts of damage by the total area of the targets $2 \times 3 + 4 \times 3 = 18$.

Tables have been introduced below, in which we have entered absolute and relative amounts of damage with respect to the sum area of the targets.

Matrix of Absolute Amounts of Damage

Number of Bombings	Target Number 1	Target Number 2
1	1,284	2,160
2	2,322	4,020
3	3,072	5,340
4	3,690	6,480

A matrix of relative amounts of damage (relative to total target area)

Number of Bombings	Target Number 1	Target Number 2
1	0,071	0,120
2	0,129	0,223
3	0,170	0,297
4	0,205	0,360

We solve the problem by a method of sorting the possible variants of target-assignment. The different variants, distinguished by the quantity of shells used against the targets, are entered in the table below.

Variant Number	Target Number 1	Target Number 2
I	0	4
II	1	3
III	2	2
IV	3	1
V	4	0

Using the matrix of relative amounts of damage and the table of different possible variants, we may determine the average total relative amounts of damage.

$$M_{I} = 0,360;$$
$$M_{II} = 0,071 + 0,297 = 0,368;$$
$$M_{III} = 0,129 + 0,223 = 0,352;$$
$$M_{IV} = 0,170 + 0,120 = 0,290,$$
$$M_{V} = 0,205.$$

We enter these results in a table

Variant Number	Total Relative Damage
I	0,360
II	0,368
III	0,352
IV	0,290
V	0,205

Therefore, the optimum target-assignment corresponds to Variant II, where one device is aimed at the first target, and three at the second.

We would have arrived at the same results, if we had used the matrix of absolute amounts of damage.

Variant Number	Total Absolute Damage
I	6,480
II	6,624
III	6,342
IV	5,232
V	3,690

From this, and the preceding table, it is apparent that the variants of destroying both targets are arranged, by their effectiveness, in the following order: II-I-IV-V.

We will consider a variant of the target assignment problem, when for each target there is an assigned, necessary, relative amount of damage, and all targets are lined up by their importance. Here, it is considered that each target is destroyed by one one type of weapon, and dispatching weapons to destroy less important objectives is not allowed until weapons have been assigned for the destruction of more important targets.

To solve the problem, we must, in advance, determine the necessary number of launchings of each weapon (considering reliability and enemy counterfire) to inflict the desired relative amount of damage to each target. After this, a matrix may be set up, whose elements are the number of launchings inflicting the assigned damage on the targets.

Type of Weapon	No. of Weapons	Target Number 1	2	3	. .	K
1	N_1	n_{11}	n_{12}	n_{13}	. . .	n_{1K}
2	N_2	n_{21}	n_{22}	n_{23}	. . .	n_{2K}
. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .
m	N_m	n_{m1}	n_{m2}	n_{m3}	. . .	n_{mK}

The elements of the matrix signify (for instance): n_{11} - the number of weapons of the first type, needed to inflict the assigned damage on the first target; n_{m2} - the number of weapons of the m-th type, needed to inflict the assigned damage on the second target, etc.

First, it is necessary to consider all of the weapons and only one target (the first is the most important). In the matrix, we must sort out (underline) all elements that do not satisfy the given indications. This sorting out occurs when:

-- for a given weapon the target is unattainable,

-- the weapons cannot fulfill the mission in the given time,

-- the weapon will not secur the safety of its own forces,

-- etc.

The logic of exclusion may be considered in the program to solve a target-assignment on an electronic calculator.

After this, we separate the elements that pertain to the second, third, fourth, fifth targets, etc. in order of decreasing target importance.

The target-assignment problem is solved in the sequence, which will be shown in the example.

<u>Example Thirty-Five:</u> Find the optimum target-assignment for two types of weapons and three targets. There are four shells of the first type and three shells of the second types. The average relative amounts of damage, which we wish to obtain, (in respect to the area of the corresponding targets), will be assigned as follows:

TABLE A

Target No.	1	2	3
Assigned Relative Damage	0,7	0,5	0,3

The number of times we must fire at the target and the types of destructive devices are given in Table B.

TABLE B

Type of Destructive device	No. of Shells	Target No. 1	2	3
1	4	6	4	2
2	3	3	2	1

We will examine the column for the first target.

TABLE C

Type of Destructive device	No. of Shells	Target No. 1
1	4	6
2	3	3

We conduct the sorting out. Since there are four shells of the first type, then, in using weapons of the first type, we must lay out prohibitions. Insofar as there is no choice, we should assign three shells of the second type for the first target.

In Table C, we underline the first column and change the quantity of shells of the second type by 3-3 = 0. We obtain Table D.

TABLE D

Type of Destructive device	No. of Shells	Target No. 2	3
1	4	4	2
2	0	2	1

We examine the column for the second target

TABLE E

Type of Destructive device	No. of Shells	Target No. 2
1	4	4
2	0	2

From Table E it follows that, insofar as there is no choice, all four shells of the first type must be aimed at the second target

The target-assignment is completed. In examining the example, it will be as follows: three shells of the second type are aimed at the most important target, and fours shells of the first type are aimed at the target that is second in importance. The third target (in importance) is not fired at, since there are not enough destructive devices.

In it's final form, the matrix of target-assignment(firing chart) will be as follows:

Type of Destructive device	No. of Shells	Target No. 1	2	3
1	4	0	4	0
2	3	3	0	0

In actuality, the quantity of targets and of weapons sent to destroy them can be much larger.

To solve the optimum target-assignment problem for many weapons and many targets in the usual way is impossible. Such problems must be solved with the aid of electronic calculators.

Calculations for Planning Combat Operations to Destroy Ships

In planning operations against ships, calculations are made to plan the needed array of weapons, to destroy the basic forces of an enemy group of ships.

These calculations, for whose production one widely applies the methods of the theory of probability, can be broken down into the following steps:

-- choosing the best route of interception for airplanes (or submarines) from their base to the region of the enemy ship's location.

-- determining the optimum distribution for the aerial (or

submarine) strike group among the elements of the ship formation, considering the relative importance of these elements.

These calculations allow one to determine:

-- the most useful region for making the strike,

-- the optimum variant of aerial (or submarine) interception,

-- the distribution of aerial (submarine) strike forces,

-- losses to aviation (submarines) enroute and in making strikes on objectives at sea.

We will examine approximate formulas, which can be used for calculations in planning VMF (Vo$_z$-dushnay Morskoy Flot) (naval air force weapons) for a battle with an enemy ship.

The route of serial (submarine) interception must be planned with consideration of:

-- the time to inflict the strike,

-- the tactical operational radius of the aerial (submarine) group,

-- the possibilities of enemy anti-aircraft (anti-submarine) defenses along the route.

We will start by considering the possibilities of enemy PVO (Protivovozdushnaya Oborona) (anti-submarine defenses).

The effectiveness of the enemy's PVO (PLO) can be evaluated by the probability of reaching the objective for an aerial (submarine) group.

The probability of reaching the target is determined by the formula

$$W_{\text{x}}^{\text{o}} = (1 - W_1)^{l_1}(1 - W_2)^{l_2} \ldots (1 - W_n)^{l_n},$$

Key: a-d.

---

where W_d is the probability of reaching the target for aircraft (submarines),

$W_1, W_2, \ldots, W_n$ are the probabilities of the airplanes (submarines) being destroyed by the enemy PVO (PLO) weapons in a single attack.

$i_1, i_2, \ldots, i_n$ are the number of attacks on each airplane (submarine) by different PVO (PLO) weapons.

The mathematical expectation of the number of planes (submarines) breaking through some line of defense will be equal to

$$M = NW_{d} = N(1 - W_1)^{i_1}(1 - W_2)^{i_2} \ldots (1 - W_n)^{i_n},$$

Key: a - d.

where M is the mathematical expectation of planes breaking through the line of defenses of PVO (PLO).

N is the number of planes (submarines) in the group, including supply units.

If one must expect to encounter new enemy PVO (PLO) forces on the route of intersection, then the mathematical expectation of the number of airplanes (submarines) that break through to the objective (last PVO' , ship's own defense) must be determined sequentially, calculating the probability of reaching the target after a certain boundary, then multiplying it by the mathematical expectation of the number of planes breaking through the preceding boundaries.

Example Thirty-Six: To strike at a convoy of ships, a group of naval rocket-carrier aviation of 80 airplanes is dispatched. On the chosen route of the flight, we may expect two lines of enemy PVO. Participating in attacks at the first line, we may expect:

-- up to 60 fighters, with a probability of destroying one of our planes in one attack of 0. 2.

-- up to 20 AA rockets with a probability of shooting down one of our planes in one attack of 0. 6.

On the second line one may expect:

-- up to 100 fighters with a probability of shooting down one of our planes of 0. 2,

-- up to 50 AA rockets; some probability of 0. 4.

Determine the mathematical expectation of the number of airplanes, breaking through to the convoy to make the strike.

Solution:

1. We calculate the average number of fighter and AA rocket attacks on our planes at the first line.

$$i_1 = \frac{60}{80} = 0{,}75; \quad i_2 = \frac{20}{80} = 0{,}25.$$

2. We calculate the probability of reaching the target (the probability of breaking through the first line of PVO):

$$W_{1a} = (1 - P_1)^{i_1} (1 - P_2)^{i_2} = (1 - 0{,}2)^{0{,}75} (1 - 0{,}6)^{0{,}25} = \\ = 0{,}86 \cdot 0{,}79 = 0{,}68.$$

Key: a - d.

3. We calculate the mathematical expectation of the number of planes breaking through the line of PVO:

$$M_1 = NW_{1a} = 80 \cdot 0{,}68 = 54.$$

Key: a - d.

4. We calculate the average number of attacks by fighters and AA rockets against the remaining aircraft at the second line:

$$i_1' = \frac{100}{54} = 1{,}85; \; i_2' = \frac{50}{54} = 0{,}92.$$

5. We calculate the probability of reaching the target (the probability of prnetrating the second line of PVO):

$$W_{2a} = (1 - P_1)^{\nu} \cdot (\quad)^{\nu} = (1 - 0,2)^{1,85} (1 - 0,4)^{0,92} =$$
$$= 0,66 \cdot 0,88 = 0,58.$$

Key: a - d.

6. We calculate the mathematical expectation of the number of airplanes penetrating the second line of PVO:

$$M_2 = M_1 W_{2a} = 54 \cdot 0,58 = 31.$$

Key: a - d

Therefore, 31 airplanes (59%) out of the group will reach the area where the convoy is located.

Analogously, we may determine the number of submarines, breaking through lines of anti-submarine defense.

Sorting all of the possible routes of the flight and the different arrangements of combinations and units, we may choose the optimum variant for the attack.

We will consider the optimum variant for the attack (or interception) to be the one for which:

-- all demands are met for making the strike at the assigned moment,

-- the mathematic expectation of the number of airplanes (or submarines) breaking through the defense lines is at maximum value.

The next step of the calculation is determining the damage, inflicted on an objective by aerial (submarine) units with different armaments, and finding the optimum plan for distributing the striking aircraft among the ships, considering the relative importance of the objectives.

Because of the large number of variants, which must be examined, problems of this kind must be solved on electronic computers.

A Probability Evaluation of the Time Spent on the Guidance Cycle

In Chapter I, conditions were introduced, characterizing the operativeness of guida .

$$T_{упр}^{a} < T_{крит}^{b} - T_{действ}^{c}$$

Key: a - guid. b - crit; c - oper.

where T_{guid} is the time spent on the guidance cycle,

T_{crit} is the critical time,

T_{oper} is the time needed by the forces to fulfill their mission.

In its turn:

$$T_{упр}^{a} = T_1 + T_2 + T_3,$$

Key: a - guid.

where T_1 is the expenditure of time in collecting information,

T_2 is the expenditure of time on reaching a decision (working out the information), and

T_3 is the time to transmit the information to the forces.

In making practical measurements of the time expenditures listed above for any actual guidance cycle, we invariably receive a scattering of values, since the continuousness of each process depends on chance factors.

To increase the reliability of our calculations in evaluating the operativeness of a guidance system, we must introduce averaged characteristics into the calculations and consider magnitudes of possible deviations from them in separate cases.

To evaluate the time T_1 and T_3 it is convenient to use the magnitude of the mathematical expectation (average value) of these times, which determine the frequency with which these or other values will appear in our chronometry study. Thus, for instance, if, in the

results of ten measurements of time T_1, we receive 1 min. in four cases, 2 min. in one case, and 3 min. in five cases, then, assuming the probability of these values appearing to be proportional to the resultant frequency, we arrive at the following law of distribution of times T_1:

Values of T_1, min.	1	2	3
Probability of these values	0,4	0,1	0,5

The mathematical expectation of the time T_1, equal to the sum of the product of each of these values and its probability, will, in this case, be equal to

$$M_{T_1} = 1 \cdot 0{,}4 + 2 \cdot 0{,}1 + 3 \cdot 0{,}5 = 2{,}1 \overset{a}{\text{мин.}}$$

Key: a - min.

The average quadratic deviation of time T_1 from its average value is equal to

$$\sigma_{T_1} = \sqrt{(1-2{,}1)^2 \cdot 0{,}4 + (2-2{,}1)^2 \cdot 0{,}1 + (3-2{,}1)^2 \cdot 0{,}5} \approx \approx 0{,}943 \ \text{мин.}^{a}$$

Key: a - min.

What time T_1 must enter the calculation? To answer this question, we must bear in mind that the time T_1 can be larger than its mathematical expectation:

-- by σ_{T_1} (i. e. be equal to 2. 1+ 0. 943≈3 min) in approximately one case out of six,

-- by $2\sigma_{T_1}$ (i. e. be equal to 2. 1+ 2 · 0. 943≈ 4 min) in approximately two or three cases out of 100,

-- by $3\sigma_{T_1}$ (i. e. be equal to 2. 1 + 3· 0. 943≈ 5 min) in one or two cases out of 1000.

But on the other hand, the time T_1 can be less than 2. 1 with the same probability. In this connection, as a rule, we can use the mathematical

expectation of the time T_1 in our calculations, i.e. the value 2.1 min. And only in separate, especially important cases will we consider its maximum value to be equal to 3, 4, or even 5 min.

To evaluate time T_2 and T_{oper}, we may use another method of substitution, according to which the expected time to make a decision (operation) is determined by the average of three other valuations: the minimum (optimistic) T_o, the maximum (pessimistic T_p (these are the values that may be met in very favorable or very unfavorable circumstances, but not more often than 1% of the cases), and the most probable T_v (the one most often met in practice). Then the expected time to make a decision (or operation) will be equal to

$$T = \frac{T_0^a + 4T_B^b + T_\Pi^c}{6},$$

Key: a - o; b - v; c - p.

and the average quadratic deviation is

$$T = \frac{T_\Pi^a - T_0^b}{6}.$$

Key: a - p; b - o.

Thus, for instance, if in very favorable circumstances the staff is able to prepare a solution to a given problem in 10 min, in very unfavorable circumstances - in 1 hr. 40 min., but usually settles the matter in 40 min., then the expected time to prepare the decision will be $T = \frac{10 + 4\cdot 40 + 100}{6} = 45$ min. Here the average quadratic deviation will be equal to $\sigma_T = \frac{100 - 10}{6} = 15$ min.

This means that in one case out of six the staff will need $45 + 15 = 60$ min; in two or three cases out of 100, it will need $45 + 2 \cdot 15 = 1$ 15 min; and in one or two cases out of 1000 $45 + 3 \cdot 15 = 1$ hr. 30 min. may be spent.

As a rule, we may use the expected time in our calculations (in the example, it is equal to 45 min.). And only in special, very important cases must we expect it to reach 1 hr., 1 hr. 15 min., and even 1 hr. 30 min.

Obtaining, by one method or another, values of time expenditure T_1, T_2, T_3, and their sum T_{gui}, and also, values T_{crit} and T_{oper}, we may establish how well each guidance organ solves different problems for each concrete instance. We may take the necessary measures to increase guidance operativeness.

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CHAPTER IV

Applying Methods of Mathematical Programming in Military Affairs

The Subject of Mathematical Programming

Mathematical programming is the totality of the mathematical methods designated to solve problems of optimum planning of guiding processes.

By guiding processes we mean processes of human activity, through which we exert different kinds of influence.

Modern science gives a lot of attention to questions of planning in all spheres of human activity: in industry, in transport, in agriculture, etc. Of exceptionally great significance is planning the combat operations of troops, which consists of rationally using forces and weapons (means) in combat and in operations.

In its most general form, the problem of optimum planning may be presented in the following manner. Suppose one must organize certain measures or a series of measures, in pursuit of a determined goal. Such goal oriented measures are sometimes called "operations".

The problem consists of planning an operation in such a way that it will be most effective, i. e. that it answers the problem assigned to it in the best way.

To give the assigned problem of optimum planning a concrete, quantitative character, we must introduce a criterion to evaluate the success and effectiveness of the operation.

The criterion of effectiveness, as indicated in Chapter I depends on the character of the operation, its goals, and its composition and is chosen for each concrete instance.

The problem of optimum planning will be contained in the choosing of a means of organizing activities, for which the criterion of effectiveness of the operation would have a minimum or maximum. If we are interested in increasing the numerical value of the criterion of

the criterion of effectiveness (for instance, the number of destroyed enemy objectives), then we must find the maximum of the criterion.

If, as the criterion of effectiveness, we have taken a quantity whose decrease is useful to us (for instance, the time to transport troops to a combat area), then we must find the minimum of this criterion. The problem of minimizing a criterion easily merges with the problem of maximizing it, for example, by reversing the sign of the criterion.

Modern mathematics deploys a whole row of methods, allowing one to solve problems of optimum guidance, which can be united under one name; "methods of mathematical programming."

The name "programming" points to the fact that the methods allow one to find a "program", a plan of action, and to change from one program to another, better one.

Methods of mathematical programming can be conditionally divided into classical and nonclassical.

To classical methods, observed in the course of mathematical analysis, belong methods of finding the maximum and minimum of functions, and also ca

To non-classical methods, developed in the last 20 or 30 years, belong: linear, non-linear, integral, stochastic, and dynamic programming.

The method of finding the maximum and minimum of a function has an exceedingly wide application in problems of optimum guidance, including military problems.

Its essence is contained in the fact, that to find the maximum and the minimum of a criterion that is a function of many variables (in the general case), one finds the derivatives of this function, for all variables, and sets them equal to zero. From the resultant system of equations one determines the values of the variables (and these in turn provide the optimum guidance), for which the maximum or minimum of the criterion is reached.

However, this method is not always applied to solve problems of optimum guidance, especially in many military problems for the following reasons.

In the first place, if there are many variables upon which the criterion depends, then such a method of finding the optimum becomes very clumsy.

In the second place, this method does not guarantee that a solution will be found. It is known that having a derivative of zero does not signify that the function is maximum or minimum. One must make an additional check, calculating second derivatives, which complicates the problem even more.

In addition, this method does give the possibility of finding the maximum (or minimum) if it lies not within, but on the border of areas of possible values of variables. Such a case is the general class of problems of optimum guidance which merge with lincar programming.

Finally, we must add, that in a number of practical cases, especially in solving military problems, the criterion of effectiveness in general cannot be differentiated (derivative dctermined), because independent variables can be continuous quantities, and discrete.

The calculus of variations is a method of finding the maximum and minimum of functionals.

By functional, we mean a function which depends itself on another function or functions.

The calculus of variations allows one to solve complicated problems of optimum guidance, in which the criterion of effectiveness depends not on a collection of variables, but on a collection of functions.

However, this method is also limited in its application in practice for essentially the same reasons as the method of finding the maximum and minimum of a function.

Linear programming is a method of finding an optimum solution when the criterion of effectiveness is a linear function of independent

variables, and the limits, imposed on the variables, are also expressed as linear dependencies.

At the present time, this method has received widespread use in solving many military problems. In another chapter this method is examined more closely, using examples for illustration.

Non-linear programming is a method of finding optimum solutions to problems in which the criterion of effectiveness and the limits of the problems are expressed by non-linear dependencies on the variables. This method is more complicated and at the present time is being developed rather strenuously. Many problems, in planning the application of weapons, for instance when several weapons are applied to one target, become problems of non-linear programming.

Integral programming is the method of finding the optimum solution to a problem in whole numbers, that is, when the hoped-for solution, corresponding to the maximum or minimum of the criterion of effectiveness, is determined by the total of variables, which are whole numbers.

In solving a problem of optimum guidance, when the variables are whole numbers, diffe.ent methods of mathematical programming are used, but with a defined specification. This specification served as a nudge to developing a special method of integral programming. Many military problems, such as problems of target distribution of anti-aircraft defense weapons, ground weapons of destruction and others demand the method of integral programming.

Stochastic programming is a method of finding the optimum guidance in operations in which chance factors play an essential role.

In such problems, the guidance process is not only determined by the initial state of the process and the chosen means of guidance, but depends also on chance, insofar as certain variables are of a random character. For this reason, this method of finding the optimum solution became known as the stochastic (or probability) method. Certain problems of stochastic programming are solved by examining only the average characteristics (mathematical expectations) of the process.

Since the pattern of the random process (which is averaged, not random, and determined) changes in advance, before the problem is solved, this method is approximate, and not always applied very extensively.

It gives good results only in those cases where the guidance system consists of a sufficiently large number of objectives (for instance, a system of anti-aircraft defense weapons). In many stochastic problems of planning this method cannot be applied, insofar as in single cases it gives too great an error, and in others it is generally useless.

At the present time there are still other special methods of approaching such problems that are just being developed

Dynamic programming is a method of finding the optimum step-by-step planning for a process of many steps, when for each step of the planning there is only one optimum step in the process.

The essence of this method is that in searching for the optimum guidance, the operation being planned is divided into a number of sequential steps, or stages. Correspondingly, the process of planning acquires many stages and developes sequentially, from stage to stage, whereas each time the guidance is optimized by one step. One must bear in mind that at each step the guidance must be chosen with respect to its previous and future states.

Dynamic programming is far-sighted and perspective. In application to planning military operations, dynamic programming allows one to make quantitative recommendations for distributing forces and weapons (means) by stages (tasks) of an operation, in such a way that the operation will be most effective.

To illustrate how a problem of dynamic programming is put, we will consider the following example. Suppose we are planning a fighter-bomber attack to destroy enemy anti-aircraft weapons. The enemy anti-aircraft weapons are in echelons of several parallel lines. Before choosing a given line, the fighter bombers pass through a zone where they are subject to attacks from the anti-aircraft weapons of that zone. The weapons of a given line may conduct fire not only at fighter-bombers attacking targets on that line, but at fighter-bombers

attacking targets beyond other lines. A fighter-bomber attack comes in sequential waves: The first wave attacks the anti-aircraft weapons of the first line, the second wave attacks targets on the second line, etc. The first wave crosses the anti-aircraft zone of the first line, and carries several losses, after which the remaining fighter-bombers attack the anti-aircraft weapons of the first line, as a result of which part of these weapons is destroyed.

In the attack of the first wave, the first line of defense is partly suppressed. Then the second wave of fighter-bombers attacks. It passes the partly suppressed first line, loses some of its planes there, enters the anti-aircraft zone of the second line, again losses some of its airplanes. The remaining fighter-bombers strike against the anti-aircraft weapons of the second line, etc.

The problem of planning the attack can be set up in the following manner: distribute the fighter-bombers in waves in order to maximize the average number of anti-aircraft weapons destroyed on all lines.

This problem has been set up under the assumption that the goal of the combat operation is to destroy the enemy anti-aircraft weapons. The criterion of effectiveness is the average number of enemy anti-aircraft weapons destroyed.

We may consider another problem where bombers must overcome an echelon system of anti-aircraft defense, in order to deliver a strike against enemy troops and objectives, located in the rear. To successfully fulfill this basic mission, some of the bombers are sent to suppress the enemy anti-aircraft weapons.

As a criterion of effectiveness of such military operations, it is useful to choose the average number of bombers that cross all of the lines of anti-aircraft defense and are ready to fulfill further combat operations.

This problem may also be solved by the method of dynamic programming.

The method of dynamic programming may be applied to evaluate prospective systems of armaments for whose creation different type of resources will be used.

We will lay out schematically an approach to solving a problem by the method of dynamic programming.

It was pointed out earlier that in the process of step-by-step guidance planning, each step must be made with consideration of future ones There is an exception to this rule. The last step is the only one that may be planned for the greatest usefulness as such.

Having planned the last step in the best manner, we may "attach" to it the next to last, and to that we may "attach" the next one in line, and so on.

Solving a problem by the method of dynamic programming is conducted in a reverse-time order; from the end of an operation to the beginning. One must note that the solutions of practical problems by the method of dynamic programming will allow one to obtain important quantitative recommendations for organizing operations. This method is of exceptionally great significance in military affairs, because many military operations are step-by-step processes. In solving complex problems of dynamic programming, demanding a large number of calculations, electronic calculating machines are used, with much success.

At the present time, however, the most practical application is that of linear programming, to which more of this chapter is devoted.

The Method of Linear Programming

The method of linear programming, which appeared about two decades ago, is widely used at the present time in questions of organizing and planning production. Recently, this method has been given a wider application in military affairs. Linear programming is used to solve problems in which certain weapons must be assigned in the best fashion, with certain limits laid down.

In application to military questions, the method of linear programming enables one to solve a number of tactical, operational and other problems in the presence of many mutually related factors.

For this, one of the quantities under analysis (for instance the criterion of effectiveness) and also the conditions of the problem are expressed as functions, dependent on a number of variables.

Then a maximum (or minimum) value of this criterion is found, and corresponding values for the variables.

These variable quantities can be, for instance: TNT equivalents, the number of anti-aircraft defense weapons to deflect an attack, the number of troops and material awaiting transport, and other important exponents of military activity.

In the results of the solution, we obtain such values for the variable quantities that we may obtain optimum distributions of weapons (the greatest effectiveness of military operations).

In problems of linear programming, the conditions which are imposed on the variables (limits of explosives, limits of anti-aircraft weapons, etc.) are determined by a system of (linear) inequalities or equalities of the first degree, where the function whose maximum (or minimum)values are being found, is a linear function with the same variables. This fact is underlined by the name "linear programming".

In analyzing practical problems, for instance problems of optimum distribution of military weapons in the course from one solution to the next one must make corresponding sequential examination of different plans (programs) of weapon distribution. Here, the name "linear programming" originates.

The practice of solving problems with linear programming has shown that for a large number of variables we must use an electronic calculator. In this case a problem which a man will spend months on can be solved in several minutes on the machine. For a large number of variables, these problems can only be solved with the help of the machine.

To solve simple problems with small numbers of variables, we may get along without even the simplest of calculating devices.

To clarify the essence of the method of linear programming, we will consider several problems, met in tactical-operational calculations.

The Problem of Transport

At two dispatching stations are concentrated, correspondingly, a_1 and a_2 tons of fuel (Fig. 17). This fuel is needed at three assigned points B_1, B_2 and B_3; and b_1, b_2, and b_3, tons of fuel must be sent to each one.

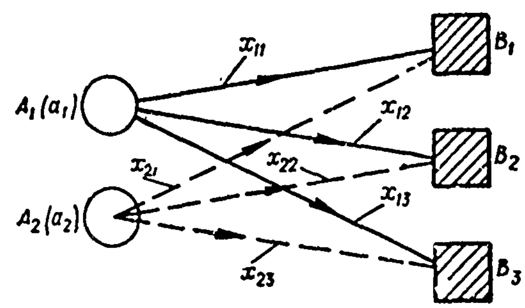

Fig. 17. Scheme of transporting loads

The cost of transporting one ton of fuel from any dispatch point to any receiving point is considered to be known. We must make a plan, so that the overall cost of transport will be the least.

We indicate by x_{ij} the number of tons of fuel to be sent from point A_i to point B_j. Then the amount of fuel to be sent from points A_1 and A_2 to point B_1 will be:

$$x_{11} + x_{21}.$$

But since we must have b_1 tons at pont B_1, we have the equation

$$x_{11} + x_{21} = b_1.$$

Analogous reasoning leads to the formulas:

$$x_{12} + x_{22} = b_2;$$
$$x_{13} + x_{23} = b_3.$$

Furthermore, we have limits, because the fuel sent from points A_1 and A_2 must be equal to the amounts stored at these points

$$x_{11} + x_{12} + x_{13} = a_1;$$
$$x_{21} + x_{22} + x_{23} = a_2.$$

These correlations are easier to obtain if all of the quantities are entered into a table, called the transport matrix.

a.	B_1	B_2	B_3	c
A_1	x_{11}	x_{12}	x_{13}	a_1
A_2	x_{21}	x_{22}	x_{23}	a_2
d	b_1	b_2	b_3	

Key: a - Dispatch points; b - Receiving Points;
c - Reserves; d - Needs.

The overall cost of transporting the fuel will be equal to

$$C = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23},$$

where C is the overall cost of transporting the fuel,

c_{ij} is the cost of transporting one tone of fuel from point A_1 to point B_j,

x_{ij} is the amount of fuel, designated from point A_i to point B_j.

As a result, the mathematical formulation of the transport problem (by criterion of cost) can be presented in the following form:

We are given a system of five linear algebraic equations with six unknowns:

The geometrical method is simple enough, but, unfortunately, it can be applied in practice only when the number of unknowns is two or three. For a large number of unknowns, the geometrical method is exceedingly complicated.

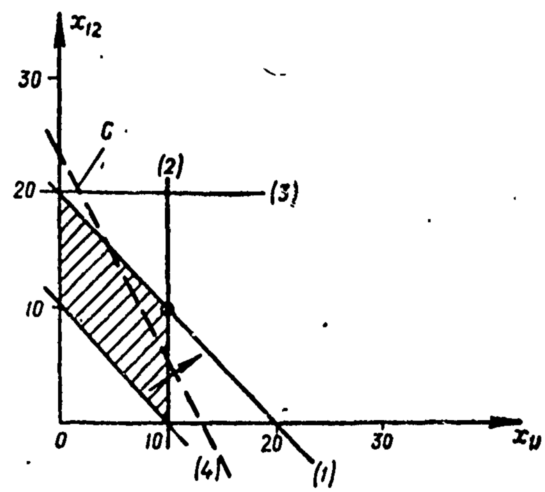

Fig. 18. Polygon of Solutions for the Transport Problem

We examine the above problem by the geometrical method. (Fig. 18). We choose a system of rectangular coordinates with axes x_{11} and x_{12}. Limits are written down (1) - (6); if we place only equal signs in them, they form geometrically straight lines. In Fig. 18 all the straight lines, expressed by the equations, are constructed, and for every straight line there is a corresponding number. The intersection of the lines forms a polygon, which is shaded in on the figure. This polygon is called the polygon of solutions of the given system of inequalities.

In the illustration there is a straight line, expressing the dependency of the overall cost C on x_{11} and x_{12} (for C = 128).

This straight line is the geometrical location of points for which the cost C has the assigned value. By changing the value of C, we obtain different lines, but they are all parallel to one another. In changing from one straight line to another, the value C changes. The arrow on the illustration indicates the direction in which we move, when going from larger values to smaller ones.

$$x_{11} + x_{21} = b_1;$$
$$x_{12} + x_{22} = b_2;$$
$$x_{13} + x_{23} = b_3;$$
$$x_{11} + x_{12} + x_{13} = a_1;$$
$$x_{21} + x_{22} + x_{23} = a_2.$$

In addition, we are given a linear function, the cost of transport:

$$C = \sum_{i,j} c_{ij} x_{ij}.$$

We must set up a transport plan so that the overall cost of transport will minimum.

For observation, we assign numerical values to the known quantities, which we enter into the table. In the center boxes we write costs of transport in rubles per ton.

a Пункты отправления	b Пункты назначения			c Запасы, тыс. m
	B_1	B_2	B_3	
A_1	4	9	3	20
A_2	4	8	1	30
Потребности, тыс. m d	10	30	10	50

Key: a - Dispatch points, b - Receiving points,
c - Reserves 1000 t., d - Demands, 1000 t.

Then the system of limits and overall cost of transport takes the form

$$x_{11} + x_{21} = 10;$$
$$x_{12} + x_{22} = 30;$$
$$x_{13} + x_{23} = 10;$$
$$x_{11} + x_{12} + x_{13} = 20;$$
$$x_{21} + x_{22} + x_{23} = 30.$$

We take x_{11} and x_{12} as independent unknowns, and express the others in terms of them[1]

$$
\begin{aligned}
x_{13} &= 20 - x_{11} - x_{12};\\
x_{21} &= 10 - x_{11};\\
x_{22} &= 30 - x_{12};\\
x_{23} &= -10 + x_{11} + x_{12}.
\end{aligned}
$$

The overall cost of transport is expressed in independent unknowns, thus:

$$C = 330 - 2x_{11} - x_{12}.$$

Since we have a solution for the system for non-negative values of the unknowns, we obtain the following inequalities:

$$
\begin{aligned}
20 - x_{11} - x_{12} &\geqslant 0; && (1)\\
10 - x_{11} &\geqslant 0; && (2)\\
30 - x_{12} &\geqslant 0; && (3)\\
-10 + x_{11} + x_{12} &\geqslant 0; && (4)\\
x_{11} &\geqslant 0; && (5)\\
x_{12} &\geqslant 0. && (6)
\end{aligned}
$$

Our problem consists of finding values of the unknowns (quantities of transported fuel) which satisfy these inequalities and give a minimum cost of transport, i. e. the minimum of the function:

$$C = 330 - 2x_{11} - x_{12}.$$

Even the simplest problem of linear programming becomes cumbersome from the point of view of calculations. There exist several methods of solving linear programming problems: the geometrical, the simplex method, the method of reverse matrices, etc.

---

[1] As is known, a system of equations has a solution, when the number of unknowns is equal to the number of equations. In problems of linear programming the number of equations is less than the number of unknowns. Therefore, some unknowns must be expressed as independent, the number of which is equal to the difference between the number of unknowns and the number of equations.

It is obvious that the optimum solution, corresponding to the minimum cost, will occur if the line C passes through the vertex of the polygon, shown on the illustration by a small circle. In this case, the cost C will, in fact, have the smallest value, and the unknowns will satisfy the limits of the problem.

When introducing proof we will point out that the optimum solution of the problem, merging with linear programming, corresponds to one of the vertices of the polygon of solutions (in the general case - a polyhedron of solutions). In this example, the optimum solution will be: $x_{11} = 10$; $x_{12} = 10$.

The remaining unknowns are found from the limiting inequalities:

$$x_{21}=0;\quad x_{13}=0;\quad x_{22}=20;\quad x_{23}=10.$$

The minimum values of cost will be equal to

$$C_{min}=330-2\cdot 10-10=300 \text{ тыс. руб.}^{a}$$

Key: a - thou. rubles.

The optimum transport plan will be as follows:

a Пункты отправления	b Пункты назначения			c Запасы, тыс. m
	B_1	B_2	B_3	
A_1	10	10	0	20
A_2	0	20	10	30
Потребности, тыс. m d	10	30	10	50

Key: a - Dispatch points; b - Receiving points;
c - Reserves thou. t; d - Demands, thou. t

Any other plan can only lead to increased transport costs.

The Problem of Distributing Weapons on Enemy Objectives (By the Maximum Number of Targets Destroyed)

The distribution of weapons among enemy objectives is immensely important in solving an operation. If a strike is made by one weapon on each objective, and we must distribute the weapons so that the total damage, inflicted on the enemy objectives, is maximum, or that the total force used is minimum, then the problem becomes one of linear programming.

Solving a problem of optimum distribution of weapons of 80 x 100 or 100 x 150 presents great difficulty and can be fulfilled only on electronic computers.

To illustrate an application of the method of linear programming we will consider the simplest problem of distribution; two weapons for three objectives (a problem of 2 x 3).

On the line of attack there are eight targets, which can be classified as three types. To destroy them we have eight weapons of two types. To destroy them we have eight weapons of two types. The probability of destruction is given for each type of target. Only one weapon is assigned to each target.

We must distribute the weapons so that the mathematical expectation of the number of destroyed targets will be maximum.

Let the numerical data of the problem be shown in a table

a Тип боеприпасов	b Тип целей			c Количество боеприпасов
	1	2	3	
I	$x_{11}/0,7$	$x_{12}/0,9$	$x_{13}/0,8$	3
II	$x_{21}/0,3$	$x_{22}/0,4$	$x_{23}/0,6$	5
Число целей d	5	2	1	8

Key: a - Type of weapon; b - Type of target;
c - Quantity of weapons; d - Number of targets.

The mathematical formulation of the problem will take the form:

$$x_{11} + x_{12} + x_{13} = 3;$$
$$x_{21} + x_{22} + x_{23} = 5;$$
$$x_{11} + x_{21} = 5;$$
$$x_{12} + x_{22} = 2;$$
$$x_{13} + x_{23} = 1.$$

The mathematical expectation of the number of destroyed targets will be equal to

$$M = 0,7x_{11} + 0,9x_{12} + 0,8x_{13} + 0,3x_{21} + 0,4x_{22} + 0,6x_{23},$$

where x_{ij} is the quantity of weapons of the i-th type, designated for targets of type j (i = 1, j = 1, 2, 3);

0. 7; 0. 9, etc. are the probabilities of destroying targets.

We take x_{11} and x_{12} as independent unknowns, and express the remaining unknowns and the mathematical expectation of the number of destroyed targets in terms of these unknowns.

As a result, we obtain

$$x_{21} = 5 - x_{11};$$
$$x_{12} = 2 - x_{22};$$
$$x_{13} = 1 - x_{11} + x_{22};$$
$$x_{23} = x_{11} - x_{22};$$
$$M = 4,1 + 0,2x_{11} - 0,3x_{22}.$$

Going to conditional inequalities, we obtain

$$5 - x_{11} \geqslant 0; \quad (1)$$
$$2 - x_{22} \geqslant 0; \quad (2)$$
$$1 - x_{11} + x_{22} \geqslant 0; \quad (3)$$
$$x_{11} - x_{22} \geqslant 0; \quad (4)$$
$$x_{11} \geqslant 0; \quad (5)$$
$$x_{22} \geqslant 0. \quad (6)$$

The problem consists of finding a distribution of weapons among targets, so that the quantity M is maximum.

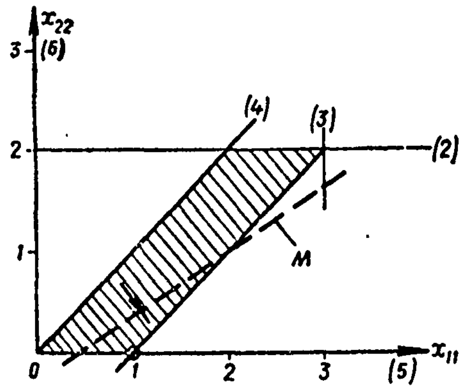

Fig. 19. A polygon of solutions for a weapon distribution problem.

In Fig. 19, we have introduced a graphic solution to this problem. In a system of coordinates of x_{11} and x_{12} are constructed straight lines, expressing the limits of the problem. From the intersections of the lines, determined by the inequalities (2), (3), (4), and (5), a polygon is obtained, any of whose points is a reachable solution, but they do not become the maximum function M. The maximum function M (shown on the graph is a dotted line) is reached when this straight line passes through the vertex of the polygon, indicated by a circle.

This vertex has the coordinates: $x_{11} = 1$; $x_{22} = 0$.

Substituting these values into the limiting equalities of the problem, we obtain the values of the remaining unknowns:

$$x_{12}=2;\ x_{21}=4;\ x_{13}=0;\ x_{23}=1.$$

Here, the mathematical expectation of the number of destroyed targets will be equal to

$$M=4,1+0,2\cdot 1-0,3\cdot 0=4,3.$$

The results of the problem solution are entered into a table.

a Тип боеприпасов	b Тип целей			c Количество боеприпасов
	1	2	3	
I	1	2	0	3
II	4	0	1	5
Число целей d	5	2	1	8

Key: a - Type of weapon; b - Type of target; c - Quantity of weapons; d - Number of targets.

Therefore, under the condition of the problem, the mathematical expectation of the number of destroyed targets will be maximum if we accept the following plan for weapons distribution:

-- one weapon of the first type is aimed at targets of the first type; two at targets of the second type, and none at the third type;

-- four weapons of the second type are aimed at targets of the first type; none at targets of the second type, and one at targets of the third type.

For any other plan of distribution, the effectiveness of the strike will be less.

We will consider another problem of optimum distribution of explosive-carrying devices among enemy objectives.

Suppose there are two types of weapons-carriers: ballistic rockets and fighter-bombers. We are assigned three types of enemy targets: rocket firing positions, command posts, and supply houses. The probabilities of destroying any of the enemy targets by rockets and by fighter bombers, and the number of carriers and targets, are given in a table. The probabilities of destroying targets with fighter-bombers have been made considering the need to overcome enemy FVO (Protivovozdushnaya Oborona) (anti-aircraft defense). For each target, we are planning to use one rocket or one fighter-bomber wing. We must distribute these carriers, so that the mathematical expectation of the number of destroyed targets will be maximum.

A table is given in the following form.

a Тип носителей		d Тип целей			h Количество носителей
		e 1 (стартовые позиции)	f 2 (КП)	g 3 (склады)	
I	b Баллистические ракеты c	$x_{11}/0{,}6$	$x_{12}/0{,}5$	$x_{13}/0{,}3$	40
II	Истребители-бомбардировщики	$x_{21}/0{,}4$	$x_{22}/0{,}7$	$x_{23}/0{,}2$	20
i	Число целей	25	15	20	60

Key: a - Type of carrier; b - Ballistic rockets;- d - Fighter-bombers; d - Type of Target; e - Rocket positions; f - CP's, g-warehouses; h - number of carriers; i - number of targets.

The quantities in the boxes indicate as follows; the numerator is the number of rockets or fighter-bomber wings, planned to destroy targets of a given type; the denominator is the probability of destroying a target. The limits of the problem may be written thus:

$$x_{11} + x_{12} + x_{13} = 40;$$
$$x_{21} + x_{22} + x_{23} = 20;$$
$$x_{11} + x_{21} = 25;$$
$$x_{12} + x_{22} = 15;$$
$$x_{13} + x_{23} = 20.$$

The mathematical expectation of the number of destroyed targets will be

$$M = 0{,}6x_{11} + 0{,}5x_{12} + 0{,}3x_{13} + 0{,}4x_{21} + 0{,}7x_{22} + 0{,}2x_{23}.$$

We choose x_{11} and x_{22} as independent unknowns, and express the remaining unknowns by the independent ones.

$$x_{21} = 25 - x_{11};$$
$$x_{12} = 15 - x_{22};$$
$$x_{13} = 40 - x_{11} - 15 + x_{22} = 25 - x_{11} + x_{22};$$
$$x_{23} = 20 - x_{22} - 25 + x_{11} = -5 + x_{11} - x_{22};$$
$$M = 0{,}6x_{11} + 0{,}5\,(15 - x_{22}) + 0{,}3\,(25 - x_{11} + x_{22}) + $$
$$+ 0{,}4\,(25 - x_{11}) + 0{,}7x_{22} + 0{,}2\,(-5 + x_{11} - x_{22}) =$$
$$= 24 + 0{,}1x_{11} + 0{,}3x_{22}.$$

Demanding non-negative unknowns, we reach the following limiting inequalities:

$$25 - x_{11} \geqslant 0;$$
$$15 - x_{22} \geqslant 0;$$
$$25 - x_{11} + x_{22} \geqslant 0;$$
$$-5 + x_{11} - x_{22} \geqslant 0;$$
$$x_{11} \geqslant 0;$$
$$x_{22} \geqslant 0.$$

We must find integral values for the unknowns which will satisfy the inequalities and yield the maximum mathematical expectation of the number of destroyed targets, (M).

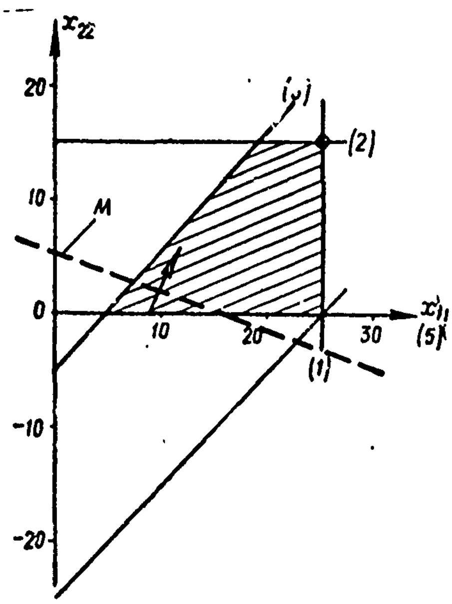

Fig. 20. A Polygon of Solutions for the Weapons-Carriers Distribution Problem

The polygon of solutions for the given problem is shown in Fig. 20. On the figure there is a straight line, expressing the dependence of M on x_{11} and x_{22} for the value of M = 25. The arrow shows in which direction the straight line moves for increased values of M. The optimum solution, corresponding to maximum M, will be at the vertex of the polygon, shown by a circle. Thus, we obtain $x_{11} = 25$ and $x_{22} = 15$.

The remaining unknowns are found from the earlier limiting inequalities:

$$x_{12}=0;\quad x_{21}=0;\quad x_{13}=15;\quad x_{23}=5.$$

The maximum mathematical expectation of the number of destroyed objectives will be

$$M_{max}=24+0{,}1\cdot 25+0{,}3\cdot 15=31.$$

For any other distribution of weapons-carriers, the enemy objectives will receive less damage.

In the given example, the optimum distribution is as follows:

-- 25 ballistic rockets must be planned for enemy firing positions, 15 for supply houses, and none for command posts.

-- 15 fighter bomber wings must be planned for enemy command posts, and 5 for enemy warehouses.

The Problem of Distributing Weapons-Carriers (By Minimum Force)

There are two types of carriers and two types of targets. There are four carriers of the first type, and two of the second type. There are two targets of the first type, three of the second type, and one of the third type. The effectiveness of the carriers is determined by the minimum amount of explosives on each one, for which the level of destruction is not less than that assigned (for instance, not less than 60%).

We assign concrete conditions to the problem in a table.

a Тип носителей	b Тип целей			c Количество носителей
	1	2	3	
I	$x_{11}/40$	$x_{12}/50$	$x_{13}/80$	4
II	$x_{21}/30$	$x_{22}/60$	$x_{23}/100$	3
Число целей d	2	3	2	7

Key: a - Type of carrier; b - Type of Target; c - Number of Carriers; d - Number of Targets.

The numbers in the boxes in the table indicate the quantity and power of the armaments in some kind of units of measure.

The limits of the given problem can be written in the form

$$x_{11}+x_{12}+x_{13}=4;$$
$$x_{21}+x_{22}+x_{23}=3;$$
$$x_{11}+x_{21}=2;$$
$$x_{12}+x_{22}=3;$$
$$x_{13}+x_{23}=2.$$

The overall power is

$$Q=40x_{11}+50x_{12}+80x_{13}+30x_{21}+60x_{22}+100x_{23}.$$

The quantities x_{ij} $(i = 1, 2;\ j = 1, 2, 3)$ indicate the quantity of armaments of the i-th type, planned to destroy targets of the j-th type.

We take x_{11} and x_{12} as the independent unknowns, and express the remaining unknowns through them.

$$x_{21}=2-x_{11};$$
$$x_{12}=3-x_{22};$$
$$x_{13}=1-x_{11}+x_{22};$$
$$x_{23}=1+x_{11}-x_{22}.$$

The overall power is expressed in terms of the independent unknowns as follows:

$$Q=360-30x_{11}-10x_{22}.$$

The condition of non-negative unknowns leads to the following limits to the problem in the form of inequalities:

$$2-x_{11}\geqslant 0;$$
$$3-x_{22}\geqslant 0;$$
$$1-x_{11}+x_{22}\geqslant 0;$$
$$1+x_{11}-x_{22}\geqslant 0;$$
$$x_{11}\geqslant 0;$$
$$x_{22}\geqslant 0.$$

We are solving the problem graphically, for which (Fig. 21) we enter straight lines on the graph, which define the polygon of solutions.

The optimum solution is located at the point where the straight line, expressing the dependence of Q on the independent unknowns, touches the vertex of the polygon of solutions (shown by a circle). The coordinates of this vertex are: $x_{11} = 0$; $x_{22} = 1$.

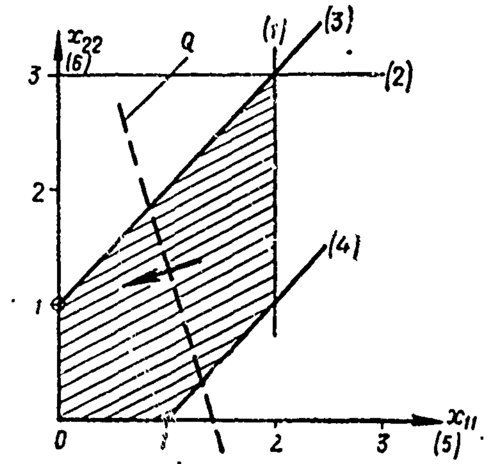

Fig. 21. A Polygon of Solutions of the Problem of Finding the Minimum Total Power

The values of the other unknowns will be:

$$x_{12} = 2;\quad x_{21} = 2;\quad x_{13} = 2;\quad x_{23} = 0.$$

The total power, which here is minimum, is equal to

$$Q_{min} = 360 - 10 \cdot 1 = 350.$$

The results of the solution of the problem are entered in a table.

a Тип носителей	b Тип целей		
	1	2	3
I	0	2	2
II	2	1	0

Key: a - Type of carrier; b - Type of Target.

Therefore, under the conditions of the given problem, the optimum distribution of carriers to targets, for which the mission is completed with a minimum expenditure of power, will be as follows:

-- the first type of carrier strikes at two targets of the second type and two of the third type;

-- the second type of carrier strikes at two targets of the first type and one target of the second type.

The Target-Assignment Problem in Anti-Aircraft Defense (PVO) Forces.

We will examine the next important problem of target-assignment in PVO forces. In repelling an attack with a limited number of weapons, each aero-space target is attacked with one PVO weapon. We must distribute the PVO weapons among the targets, so that the attack is deflected with the greatest effectiveness.

As the criterion of effectiveness we take the mathematical expectation of the number of enemy aero-space targets destroyed.

We will consider a concrete example of this problem. Suppose we have two types of PVO weapons: AA rockets and fighters. These weapons may repel an attack of three types of enemy targets.

The probability of destroying any of the enemy targets with each type of PVO weapon, and the overall quantities of weapons and targets are shown in the table.

	a Тип и вид средств	d Тип целей 1	2	3	e Количество средств
I	b ЗУР	$x_{11}/0,8$	$x_{12}/0,7$	$x_{13}/0,6$	20
II	c Истребители	$x_{21}/0,6$	$x_{22}/0,3$	$x_{23}/0,5$	40
	f Число целей	20	10	30	60

Key: a - Type and form of weapon; b - ZUR (Guided anti-aircraft rocket); c - Fighters; d - Type of target; e - quantity of weapons; f - number of targets.

The quantities in the numerators indicate the unknown quantity of PVO weapons planned for the destruction of a given type of target. In the denominator is the probability of destruction.

We will consider that each aerial target is attacked by only one PVO weapons.

The limits of the problem will take the form

$$\begin{gathered} x_{11}+x_{12}+x_{13}=20; \\ x_{21}+x_{22}+x_{23}=40; \\ x_{11}+x_{21}=20; \\ x_{12}+x_{22}=10; \\ x_{13}+x_{23}=30. \end{gathered}$$

The mathematical expectation of the number of destroyed targets is written in the form

$$M=0{,}8x_{11}+0{,}7x_{12}+0{,}6x_{13}+0{,}6x_{12}+0{,}3x_{22}+0{,}5x_{23}.$$

We choose x_{11} and x_{22} as the independent unknowns and express the remaining unknowns in terms of them:

$$\begin{gathered} x_{21}=20-x_{11}; \\ x_{12}=10-x_{22}; \\ x_{13}=20-x_{11}-10+x_{22}=10-x_{11}+x_{22}; \\ x_{23}=40-20+x_{11}-x_{22}=20+x_{11}-x_{22}. \end{gathered}$$

The mathematical expectation of the number of destroyed targets is expressed in terms of the independent unknowns in the following manner:

$$M=35+0{,}1x_{11}-0{,}3x_{22}.$$

Since the solution to the problem corresponds to non-negative values for the unknowns, we come to the following limiting inequalities:

$$\begin{gathered} 20-x_{11}\geqslant 0; \\ 10-x_{22}\geqslant 0; \\ 10-x_{11}+x_{22}\geqslant 0; \\ 20+x_{11}-x_{22}\geqslant 0; \\ x_{11}\geqslant 0; \\ x_{22}\geqslant 0. \end{gathered}$$

The problem consists of finding values of the unknowns which satisfy the system of inequalities, and give a maximum value for M.

The polygon of solutions for the given problem is as shown in Fig. 22. In the figure is a straight line, expressing the dependence of M on x_{11} and x_{22}. The arrow shows the direction in which the line moves for increasing M.

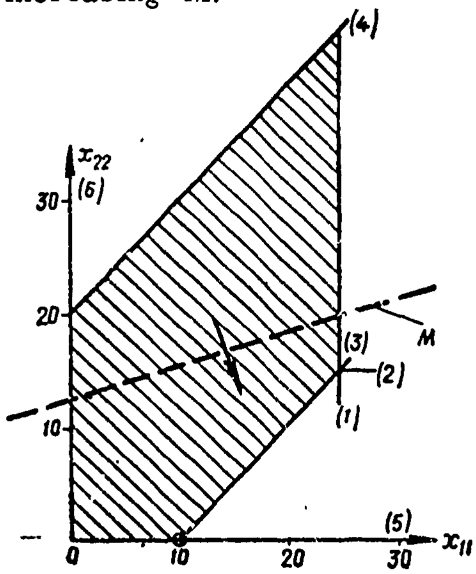

Fig. 22 - The Polygon of Solutions for the Problem of PVO Target-Distribution

The maximum of quantity M will be reached in the vertex of the polygon, shown by a small circle.

The values of the independent unknowns at this vertex will be x_{11} $x_{11} = 10$; $x_{22} = 0$.

We obtain the values of the other unknowns from the limiting inequalities: $x_{12}=10;\ x_{21}=10;\ x_{13}=0;\ x_{23}=$ [illegible]

The maximum value of the mathematical expectation of the number of destroyed enemy aerial targets will be

$$M_{max} = 35 + 0{,}1 \cdot 10 = 36.$$

Therefore, the optimum target-assignment, under the conditions given in this problem will be as follows:

-- ten AA rockets must be sent to destroy targets of the first type and ten for those of the third type;

-- ten fighters must be sent to destroy targets of the first type, none for targets of the second type, and 30 for those of the third type.

The Problem of Optimum Distribution of Naval Weapons in a Battle With Enemy Ships

We will examine the prot m of distributing naval forces in a battle with enemy ships, through an example.

There are two types of enemy ships and we have three types of weapons: submarines with torpedoes, naval rocket-carrier aviation, and submarines with ballistic missiles. The probabilities of destroying enemy targets, for each type of weapon, and the number of weapons and targets are entered in a table.

a Тип и вид средств		b Тип целей 1	2	c Количество средств
I d	ПЛ с торпедами	$x_{11}/0,4$	$x_{12}/0,6$	2
II e	МРА с крылатыми ракетами	$x_{21}/0,7$	$x_{22}/0,8$	3
III f	ПЛ с баллистическими ракетами	$x_{31}/0,3$	$x_{32}/0,6$	1
g Число целей		2	4	6

Key: a - Type and form of weapon; b - Type of target;
c - Number of weapons; d - PL (sub) with Torpedos,
e - MRA (rocket-aviation) with winged rockets;
f - PL with ballistic missiles; g - Number of Targets.

The quantities in the numerator indicate the unknown numbers of naval weapons, planned to destroy each type of target, and in the denominator are probabilities of destruction.

We will stipulate that for each target there is only one weapon.

The limits of the problem take the form:

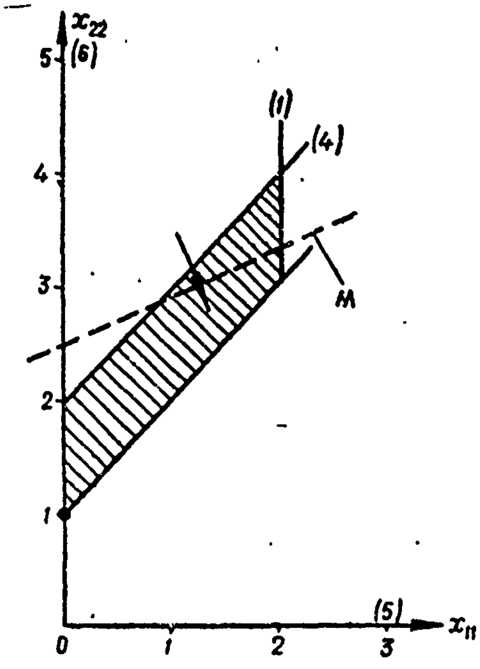

Fig 23. Polygon of Solutions for the Problem of Target-Assignment Against Enemy Ships.

$$x_{11} + x_{12} = 2;$$
$$x_{21} + x_{22} = 3;$$
$$x_{31} + x_{32} = 1;$$
$$x_{11} + x_{21} + x_{31} = 2;$$
$$x_{12} + x_{22} + x_{32} = 4.$$

The mathematical expectation of the number of destroyed ships is written in the form

$$M = 0{,}4x_{11} + 0{,}6x_{12} + 0{,}7x_{21} + 0{,}8x_{22} + 0{,}3x_{31} + 0{,}6x_{32}.$$

We take x_{11} and x_{22} as the independent unknowns, and express the remaining unknowns and the wuantity M through them.

$$x_{12} = 2 - x_{11};$$
$$x_{21} = 3 - x_{22};$$
$$x_{31} = 2 - x_{11} - 3 + x_{22} = -1x_{11} + x_{22};$$
$$x_{32} = 4 - -2 + x_{11} - x_{22} = 2 + x_{11} - x_{22};$$
$$M = 4{,}2 + 0{,}1x_{11} - 0{,}2x_{22}.$$

Since the solution of the problem corresponds to non-negative values for the unknowns, we arrive at the following limiting inequalities.

$$2 - x_{11} \geqslant 0;$$
$$3 - x_{22} \geqslant 0;$$
$$-1 - x_{11} + x_{22} \geqslant 0;$$
$$2 + x_{11} - x_{22} \geqslant 0;$$
$$x_{11} \geqslant 0;$$
$$x_{22} \geqslant 0.$$

The problem consists of finding values for the unknowns that will satisfy the system of inequalities and yield the maximum mathematical expectation for the number of destroyed enemy snips.

The polygon of solutions of this problem is shown in Fig. 23. The straight line M is shown in the figure, and the direction in which the line moves for increasing M is shown by an arrow. The

optimum solution corresponds to the vertex of the polygon with coordinates $x_{11} = 0$; $x_{22} = 1$.

Values for the remaining unknowns may be obtained from the limiting inequalities: $x_{12}=2$; $x_{21}=2$; $x_{31}=0$; $x_{32}=1$.

The maximum mathematical expectation of the number of destroyed targets will be equal to

$$M_{max} = 4{,}2 - 0{,}2 \cdot 1 = 4.$$

Therefore, in order to fulfill this mission with the most effectiveness, we must distribute the naval weapons among the targets in the following manner:

-- no submarines for targets of the first type, two for targets of the second type,

-- two winged rockets for targets of the first type and one for a target of the second type,

-- no submarines with ballistic missiles for targets of the first type, one for a target of the second type.

With such a distribution we may expect four enemy targets to be destroyed.

In any other distribution the number of destroyed targets will be less.

The Problem of Distributing Means of Observation

We will examine the following problem which is one of optimum distribution of means of observation for a given effectiveness.

There are three types of means of observation: an airplane with a photo-apparatus, and airplane with a radio direction-finding apparatus, and a submarine. There are also two types of targets which must be found by our reconnaisance. The probabilities of detecting and locating any target, for each type of apparatus, nd also

the number of apparatuses and targets are entered into a table.

a Тип и вид средств		b Тип целей		c Количество средств
		1	2	
I	d Самолет с фотоаппаратурой	$x_{11}/0,5$	$x_{12}/0,7$	4
II	e Самолет с радиолокационной аппаратурой	$x_{21}/0,6$	$x_{22}/0,5$	3
III	f ПЛ	$x_{31}/0,7$	$x_{32}/0,6$	4
g Число целей		4	7	11

Key: a - Type and form of apparatus;
b - Type of Target
c - Quantity of Apparatuses
d - Airplane with photo-equipment
e - Airplane with radio d. f. equipment
f - Submarines,
g - Number of targets

The quantities in the numerators indicate the unknown numbers of recon devices. Probabilities of determining target locations are shown in the denominators. We will stipulate that only one recon device is sent to find each target.

The limits of the problem have the form

$$x_{11} + x_{12} = 4;$$
$$x_{21} + x_{22} = 3;$$
$$x_{31} + x_{32} = 4;$$
$$x_{11} + x_{21} + x_{31} = 4;$$
$$x_{12} + x_{22} + x_{32} = 7.$$

The mathematical expectation of the number of detected targets is written in the form

$$M = 0,7x_{11} + 0,7x_{12} + 0,6x_{21} + 0,5x_{22} + 0,9x_{31} + 0,6x_{32}.$$

Taking x_{11} and x_{22} as the independent unknowns, and expressing the other unknowns and the function M in terms of them, we obtain

$$x_{12} = 4 - x_{11};$$
$$x_{21} = 3 - x_{22};$$
$$x_{31} = 1 - x_{11} + x_{22};$$
$$x_{32} = 3 + x_{11} - x_{22};$$
$$M = 7{,}9 - 0{,}5x_{11} + 0{,}2x_{22}.$$

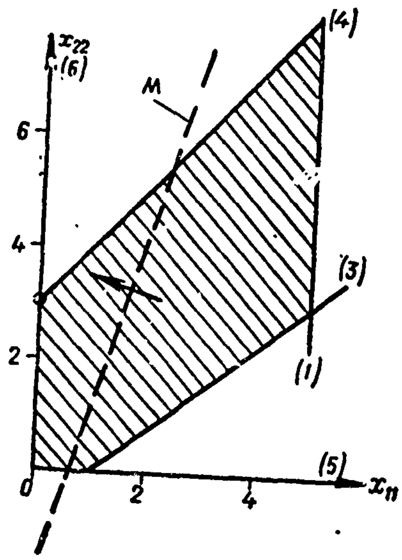

Fig. 24 - Polygon of Solutions for the Problem of Distributing Recon Devices

The condition of non-negative solutions to the problem leads to the following limiting inequalities:

$$4 - x_{11} \geqslant 0;$$
$$3 - x_{22} \geqslant 0;$$
$$1 - x_{11} + x_{22} \geqslant 0;$$
$$3 + x_{11} - x_{22} \geqslant 0;$$
$$x_{11} \geqslant 0;$$
$$x_{22} \geqslant 0.$$

The polygon of solutions for this problem is shown in Fig. 24. The direction in which the line moves for increasing M is shown by the arrow.

The optimum solution of the problem corresponds to the vertex of the polygon with coordinates $x_{11} = 0$; $x_{22} = 3$.

The values of the remaining unknowns are obtained from the limiting inequalities:

$$x_{12}=4;\ \ x_{21}=0;\ \ x_{31}=4;\ \ x_{32}=0.$$

The maximum mathematical expectation of the number of targets observed is equal to

$$M_{max}=7{,}9+0{,}2\cdot 3=8{,}5.$$

Consequently, in the conditions of this problem, the distribution of reconnaisance forces will be as follows:

-- aircraft with photo equipment will not be used to observe targets of the first type, but two will be used for targets of the second type,

-- aircraft with radio direction-finding equipment will not be used to detect targets of the first type, but three will be used for targets of the second type,

-- four submarines will be used to detect targets of the first type, but none for targets of the second type.

The Problem of Constructing a Defense System by Cost Criteria

In all of the earlier examples, the area of realizable solutions was closed, forming a four- or five-sided figure. However, there is a class of problems in linear programming, in which the area of realizable solutions is unlimited, but, as in the preceding examples, the solution to this problem is also found on a graph of this area.

We will consider the problem of constructing a PVO (Protivovozdushnaya Oborona - anti-aircraft defense) defense system, with minimum cost as the criterion.

We will assume that it is necessary to build an anti-aircraft system to defend an objective.

We do not know from what height the enemy aerial attack will come, but we may assume, that a maximum of 100 planes may attack from a low altitude, 150 from a medium altitude, and 100 from a high altitude.

$$2)\ 0{,}5x + 0{,}5y \geqslant 150.$$

This inequality expresses the demand that at medium altitudes the mathematical expectation of the number of planes shot down by low-altitude (0. 5x) and high-altitude rockets (0. 5y), is not less than 150.

$$3)\ 0{,}25x + 0{,}75y \geqslant 100.$$

This inequality demands that not less than 100 planes are shot down at high-altitudes.

In addition, there are two limits, as follows:

$$4)\ x \geqslant 0;$$
$$5)\ y \geqslant 0.$$

These inequalities express the fact that the unknown number of high- and low-altitude complexes cannot be negative.

We will transform these five inequalities, giving them a simpler and more convenient form:

$$1)\ 3x + y \geqslant 400;$$
$$2)\ x + y \geqslant 300;$$
$$3)\ x + 3y \geqslant 400;$$
$$4)\ x \geqslant 0;$$
$$5)\ y \geqslant 0.$$

If we ignore the sign "greater than", and observe only the equal sign, then all five equations, insofar as all of their unknowns are of the first degree, are equations of straight lines (linear).

We construct these lines on a graph (Fig. 25). It is evident on the graph, that the equations x = 0 and y = 0 are the axes of the coordinates,

At our disposition there are two types of anti-aircraft rocket complexes, high- and low-altitude, which have a different probability of destroying a target, depending on how high it is flying. In addition, the high-altitude rockets are twice as expensive as the low-altitude rockets.

How many and which sort of complexes should be used, so that on the one hand the expected number of destroyed targets is not less than the number of planes that may participate in an attack, and, on the other hand, the cost expenditure is kept at a minimum?

To solve this problem we enter initial figures[1] into a table

a Тип комплексов	b Вероятность поражения целей на высотах			f Стоимость ракеты (в единицах стоимости)
	c малой	d средней	e большой	
Маловысотные g	0,75	0,5	0,25	25
Высотные h	0,25	0,5	0,75	50
i Максимальное число атакующих самолетов на высотах	100	150	100	

Key: a - Type of complex
b - Probability of destroying targets by altitudes
c - low
d - medium
e - high
f - Cost of Rockets (in units of cost)
g - Low altitude
h - High Altitude
i - Maximum number of attacking aircraft by altitudes.

We indicate the unknown number of low-altitude complexes by x and the number of high-altitude complexes by y.

The limits of the problem will have the form

$$1)\ 0{,}75x + 0{,}25y \geqslant 100.$$

This inequality expresses the demand that the mathematical expectation of the number of planes shot down at low-altitudes, by low-altitude rockets (0.75x) and high-altitude (0.25y) rockets, does not total less than 100.

[1] The initial figures are of a purely illustrative character.

and the remaining three equations form three intersecting straight lines, inclined at different angles. Here, the heavy broken line ABCD defines the border of the region of realizable solutions (shaded in on the graph). This means any point on the given area will have coordinates (x, y) that satisfy the five inequalities introduced above.

However, the problem is not solved yet.

The problem is that from the entire set of these points we must find a point which will fulfill another demand, the achievement of a minimum cost for all of the rockets, expended in repelling the attack.

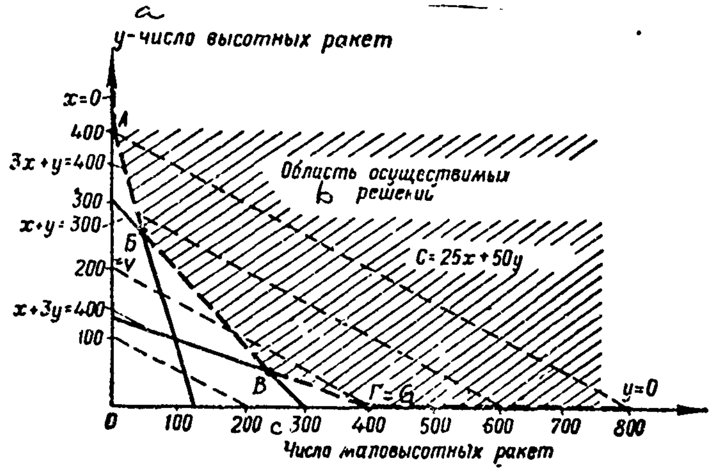

Fig. 25 - To Solve the Problem of Constructing a Defense System by Cost Criteria

Key: a - Number of High-Altitude rockets;
b - Region of realizable solutions
c. - Number of low-altitude rockets

To find this point, we form the equation which defines the cost C. The overall cost of rockets (in units of cost) is the sum of the cost of x low-altitude and y high-altitude rockets, i.e.

$$C = 25x + 50y.$$

This equation is also a straight line. Depending on the magnitude of the cost C, this line will move on the graph parallel to itself

(dotted line). Here, the lower the cost C, the closer the line is to the origin of coordinates 0.

The point, which is the solution of the problem, must satisfy two conditions:

-- it must not be outside the limits of realizable solutions,

-- it must lie on the line of minimum cost.

Obviously, the point that satisfies both conditions is the point C on the graph of realizable solutions.

The coordinates of this point (x = 250, y = 50) are also the solution to the problem. In fact, in using 250 low-altitude and 50 high-altitude rockets, we will, in the first place, secure the destruction of not less than the assigned number of targets for each altitude, and in the second place, have minimum expenditure.

The expenditures will be equal to .

$$C = 25x + 50y = 25 \cdot 250 + 50 \cdot 50 =$$
$$= 8750 \text{ единиц стоимости.}$$

a Units of cost

It may be shown that any other point in the area of possible solutions will yield a larger cost. The comparative cost of a PVO (anti-aircraft) system, (for points A, B, C, and D), are shown in the table.

a Значение величин	b Точки			
	c А	d Б	e В	f Г
x	0	50	250	400
y	400	250	50	0
C	20 000	13 750	8 750	10 000

Key: a - Quantities
b - Points
c - A
d - B
e - C
f - D

From the table it is apparent that the solution found by the method of linear programming (point C) will secure a minimum cost for the system.

The Problem of Distributing Weapons by Time Criteria

To solve this problem we use another so-called tabular method, and apply it to a conditional problem, created for illustrative purposes.

Suppose we have three military groups, with a different number of units in each one:

-- Group I - 100 units,

-- Group II - 100 units,

-- Group III - 120 units.

The problem is to destroy objectives, sequentially, in the shortest time (it is assumed that each unit is capable of reaching any objective). We will assume that in earlier calculations we found that:

For the 1st target, 40 units are needed
For the 2nd target, 80 units are needed
For the 3rd target, 80 units are needed
For the 4th target, 80 units are needed
For the 5th target, 80 units are needed.

In addition, it is known that the intervals between the application of military units, from each group, will be as follows:

For Group I, firing at the first objective -- 2 min., at the second objective, -- 1 min., at the third, -- 2 min., at the fourth, -- 3 min., at the fifth, -- 3 min.

Analogous figures pertain to Groups II and III.

With these figures, how does one find the best variant from the planning table, when the possible variants number into the millions? Linear programming allows this to be done in a limited number of tries.

First of all, we distribute the objectives among the groups, applying the most obvious program: we use all of the units of Group I, and when their limit is reached we begin to use units of the next group. As a result, we obtain the following table for the firing plan.

a 1-й вариант

b Номер группировок	d Номер объектов 1	2	3	4	5	Количество боевых единиц c
I	40^2	40^1	20^2	0^3	0^3	100
II	0^2	0^2	60^2	40^1	0^1	100
III	0^3	0^3	0^2	40^1	80^2	120
c Количество боевых единиц	40	40	80	80	80	320

Key: a - 1st variant
b - Group number
c - Number of military units
d - Objective number

The table shows how many units are launched from each group to each target. The corresponding interval between the use of different units is indicated as an exponent (index of degree). From the table it follows that:

The 1st objective is fired at for 80 min. (40 units at 2 min. each)
The 2nd objective -- 40 min. (40 units, 1 min. each),
The 3rd objective --160 min. (80 units, 2 min. each),
The 4th objective -- 80 min. (80 units, 1 min. each),
The 5th objective -- 160 min. (80 units, 2 min. each)

The total is 520 min. (8 hrs. 40 min.).

What should be done further to shorten the time expenditure?

Examining the first row of the table, we find the first zero (it is in the fourth column and shows that units from Group I do not fire at the fourth objective).

We increase the zero to one (we check to see if we can gain time if Group I sends even one unit against the fourth objective). This means that we must change the table somewhat to that the sums in the rows and columns are maintained. The table will appear as follows:

a 2-й вариант

b Номер группировок	c Номер объектов 1	2	3	4	5	d Количество боевых единиц
I	40	40	$(20-1)^2$	$(0+1)^3$	0	100
II	0	0	$(60+1)^2$	$(40-1)^1$	0	100
III	0	0	0	40	80	120
d Количество боевых единиц	40	40	80	80	80	320

Key: a - 2nd variant; b - Group Number; c - Objective Number; d - Number of military units.

We test this change. Taking one unit from Group I from the 3rd objective, we gained 2 minutes, but assigning it to the fourth objective, we lost 3 min. (total loss of 1 min.). Adding one unit from Group II to the third objective, we lose 2 min., and by taking it from the fourth objective we have gained 1 min. (total loss of 1 min.).

In all, we have obtained an insignificant result; the overall firing time has increased by 2 min.

Then we try increasing the second zero to one (first row, fifth column), while at the same time rearranging the plan so that the sums of the rows and column stay the same. We obtain the third variant of the table.

d 3-й вариант

a Номер группировок	b Номер объектов 1	2	3	4	5	Количество боевых единиц c
I	40	40	$(20-1)^2$	0	$(0+1)^3$	100
II	0	0	$(60+1)^2$	$(40-1)^1$	0	100
III	0	0	0	$(40+1)^1$	$(80-1)^2$	120
c Количество боевых единиц	40	40	80	80	80	320

Key: a - Group Number; b - Objective number; c - Number of military units; d - 3rd variant.

For Group I, the time has decreased by 2 min. and increased by 3 min. (a loss of 1 min.); for Group II, the time has increased by 2 min. and decreased by 1 min. (a loss of 1 min.); for Group III the time has increased by 1 min. and decreased by 2 min. (a gain of 1 min.).

However, in all, we have lost one minute.

We will do this experiment with the next zero, and with all of the zeroes in order, entering the results of the changes in a table.

a – Результаты единичных изменений

b Номер группировок	c Номер объектов					d Количество боевых единиц
	1	2	3	4	5	
I	40^2	40^1	20^2	0^3_{+1}	0^3_{+1}	100
II	0^2_0	0^2_{+1}	60^2	40^2	0^1_{-1}	100
III	0^3_{+1}	0^3_{+2}	0^2_0	40^1	80^2	120
d Количество боевых единиц	40	40	80	80	80	320

Key: a - Results of changing zeroes by one;
b - Group Number
c - Objective number
d - Number of military units,

The lower indices by the zeroes show the change in firing time for each time the zeroes were replaced by ones. From the table, we see only one useful change, when one unit from Group II is sent to the 5th objective instead of the fourth. This reduces the firing time by one minute. Then we use this solution by changing all 40 units of Group II from the 4th objective to the 5th (that is, we do the maximum that is possible in reducing the time).

Performing the corresponding change, we obtain the 4th variant of the table.

d 4-й вариант

a Номер группировок	b Номер объектов				c	Количество боевых единиц
	1	2	3	4	5	
I	40^2	40^1	20^2	0^3	0^3	100
II	0^2	0^2	60^2	0^1	40^1	100
III	0^3	0^3	0^2	80^1	40^2	120
c Количество боевых единиц	40	40	80	80	80	320

Key: a - Group Number; b - Objective number; c - Number of military units; d - 4th variant

The overall firing time in this case will be

Общее время стрельбы в этом случае будет равно:

$$(40\cdot 2)+(40\cdot 1)+(20\cdot 2)+(60\cdot 2)+(80\cdot 1)+(40\cdot 1)+$$
$$+(40\cdot 2)=80+40+40+120+80+40+80=480 \ \text{мин}^{a}$$

Key: a - 480 min. (8 hours)

In comparison with the original plan, we have gained 40 minutes.

Continuing with this newer and more useful table of firing, we apply the same method, that is, we change each zero to one, in turn. The results of these changes are entered into a table.

a Результаты единичных изменений

b Номер группировок	c Номер объектов					d Количество боевых единиц
	1	2	3	4	5	
I	40^2	40^1	20^2	0^3_{+3}	0^3_{+2}	100
II	0^2_0	0^2_{+1}	60^2	0^1_{+1}	40^1	100
III	0^3_0	0^3_{+1}	0^2_{-1}	80^1	40^2	120
d Количество боевых единиц	40	4C	80	80	80	320

Key: a - Results of changing zeros into ones;
b - Group Number
c - Objective number
d - Number of military units

out that we may also receive a gain here if we transfer Group III to the third objective. We transfer all 40 rockets from the 5th to the 3rd objective. We obtain a new variant of the table.

a 5-й вариант

b Номер группировок	c Номер объектов					d Количество боевых единиц
	1	2	3	4	5	
I	40^2	40^1	20^2	0	0	100
II	0	0	20^2	0	80^1	100
III	0	0	40^2	80^1	0	120
d Количество боевых единиц	40	40	80	80	80	320

Key: a - 5th variant; b - Group number; c - Objective number; d - Number of military units.

The overall firing time in this case will be:

Общее время стрельбы в этом случае будет равно:

$$(40\cdot 2) + (40\cdot 1) + (20\cdot 2) + (20\cdot 2) + (40\cdot 2) + (80\cdot 1) + (80\cdot 1) = 80 + 40 + 40 + 40 + 80 + 80 + 80 = 440 \text{ a мин}$$

Key: a - 440 min. (7 hours 20 minutes). This means we have gained 40 minutes.

Once again we change all of the zeros into ones and enter the results into a table.

Results of Changing Zeros Into Ones

a Результаты единичных изменений

b Номер группировок	c Номер объектов					d Количество боевых единиц
	1	2	3	4	5	
I	40^2	40^1	20^2	0^3_{+2}	0^3_{+2}	100
II	0^2_0	0^2_{+1}	20^3	0^1	80^1	100
III	0^3_{+1}	0^3_{+2}	40^2	80^1	0^2_{+1}	120
d Количество боевых единиц	40	40	80	80	80	320

Key: a - Group number c - Number of military units.
b - Objective number

From the table it is apparent that none of the changes leads to the desired result. Any change can only increase the firing time (or leave it unchanged).

This shows that we have obtained the optimum solution to this problem (the 5th variant of the table). In comparison with the first plan (1st variant) we have economized on time by 1 hour and 40 minutes, and for military operations, such a gain is extremely important.

The General Problem of Linear Programming in Tactical-Operational Computations

We have examined particular problems, from which it is apparent that methods of linear programming can be used to solve important practical problems of a military character.

In all cases, we have examined tables (matrices), in which the number of means (weapons), multiplied by the number of targets, is equal to six. This was explained by the fact that in such a matrix one may apply a visual and simple geometrical method, where the optimum solution is obtained from one of the vertices of the polygon of solutions. If the number of independent unknowns is equal to three, the optimum solution is obtained from one of the vertices of the polyhedron of solutions. If the number of independent unknowns is greater than three, then the problem does not have a simple geometrical

interpretation, although in these cases one does use a concept of multi-dimensional geometry.

To solve problems of linear programming for large values of m and n, such methods are applied, as the simplex-method, the method of inverse matrices, and others. However, in the case of large matrices, the solution, by any method, becomes so clumsy, that it becomes difficult to obtain it with the simplest of calculation methods. At the present time, the largest problems connected with linear programming, are successfully solved on electronic computers.

Problems of optimum distribution of forces and means in operations, can often be linked to methods of linear programming, where the number of types of weapons and objectives is significant.

The basic indicator that a problem of weapons distribution in an operation may be linked to methods of linear programming, is when each enemy target has only one weapon attached to it. If more than one weapon is planned against one target, then such a problem, as was said before, is linked to methods of non-linear programming.

The general sequence of solving proglems in which linear programming is applied to optimum weapons distribution will be as follows. First of all, we must determine the combat effectiveness (probability of destruction, relative damage, probability of detection, etc.) of each type of weapon (means) on each type of objective.

Then a matrix is formed, in which the rows correspond to the weapons, and the columns to the objectives. In each cell of the matrix is the unknown quantity of weapons and the known effectiveness of the given weapon for the given objective. Assigned quantities of weapons and targets are written in the matrix.

After this, limits to the problem are written, in the form of equalities or inequalities. Depending on the character of the problem, a criterion is chosen, which characterizes the general effectiveness of the weapons distribution (the total power of the armaments, the total relative damage, the mathematical expectation of the number of destroyed targets and others). The criterion of effectiveness must be expressed in terms of an assigned effectiveness for the separate weapons and the number of weapons assigned to each type of target. If, in the formulation of the criterion of effectiveness, and in the

equations which determine the limits of the problem, the unknowns are of the first degree, then the problem is one of linear programming. Different methods may be applied, depending on the dimensions and complexity of the matrix.

CHAPTER V

THE APPLICATION OF THE THEORY OF MASS SERVICE TO MILITARY AFFAIRS

The Theory of Mass Service

The theory of mass service investigates the quantitative side of processes related to the organization of mass service. Here, by "service", in the broadest sense of the word, we mean the functioning of some system of apparatuses, designed to fulfill massive and uniform demands.

In practical human activity, a situation is often created, where a massive demand arises for service of some special form, but the service system, deploying only a defined number of service apparatuses with limited productivity, is not always capable of satisfying all the requests. For illustration, one may introduce the tasks of telephone exchanges, airports, mooring places, gasoline filling stations, hospitals, ticket counters, repair points and other institutions of mass service.

In all similar cases, this theory solves one basic problem. It establishes, with all possible accuracy, a relationship between the number of service apparatuses (with a given productivity), and the quantity of demands arising, so that the productive capacity of the system satisfies these demands.

The significance of this theory is very great in military affairs, where there so very often arises conflict between the need to satisfy demands and the possibility of fulfilling them with a limited number of service units that make up a system.

Examples of Systems of Mass Service

We will introduce three examples of mass service of a military character, suing the following terminology.

Service System	Service Apparatus of the System	Nature of Service	Input Demands for Service	Output of Serviced Demands.
Aviation Repair Network	Aircraft repair shops	Repair of aviation equip.	Aircraft, engines, & other aviation equip. awaiting repair.	Repaired aviation equip.
Group of anti-aircraft complexes.	Anti-aircraft complexes	Destruction of enemy aircraft and rockets	Enemy planes and rockets, sighted by an observation system and entering the zone of defense.	Enemy planes and rockets shot down.
System of sanitary facilities for personnel.	Sanitary points	Sanitary facilities for personnel.	Personnel needing sanitary facilities	Personnel having used the sanitary facilities.

As is apparent from these examples, the term "service" is used here not only in the sense to which we are accustomed. If, in customary speech, service is associated with the satisfaction of certain needs, then in the theory of mass service, it acquires a wider meaning (this is apparent, in particular, from the second example in the table).

The task of the theory, is the expl in the basic quantitative characteristics of the process of mass service, allowing one to evaluate the quality of the organization of that service. Using such characteristics, one may find the weak places in the organization, and consciously work to its improvement.

As is apparent from the table, a flow of input demands enters the service system, and an output flow leaves it.

Diversity of Systems of Mass Service

The productive capacity of a system may be sufficient or insufficient. By "insufficient", we mean a system which, in a unit of time, is capable of serving less than the demands made on it.

In those cases, when the productive capacity of a system is sufficient, there is no waiting line for the system's input, and the output consists only of demands that have been fulfilled.

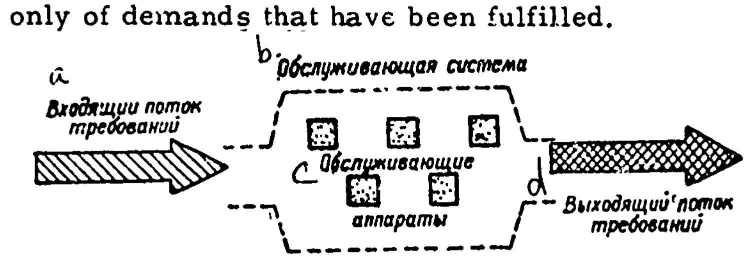

Fig. 26 - Productive Capacity of a Sufficient Service System

Key: a - Input flow of demands   c - Service apparatuses
b - Service System   d - Output flow

The basis quantitative criterion of the quality of the work of such a system is the fullness of the load (the average percent of use) of the service apparatuses. Here, a small percent of use indicates a superfluous number of service apparatuses in a given system, as a result

of which when a given number of demands arrive in a given unit of time, a significant number of apparatuses are not in use.

In cases (Fig. 27), when the productive capacity of a system is insufficient, but the demands can wait for the service to begin, (for instance, airplanes entering a repair shop) the output flow will consist entirely of serviced demands, (repaired aircraft), but here, a line of demands begins to accumulate (planes, awaiting repair).

Such a system is called a system with waiting. The basic quantitative criterion of the quality of the work of such a system is the average length of the line (the average number of demands, awaiting service).

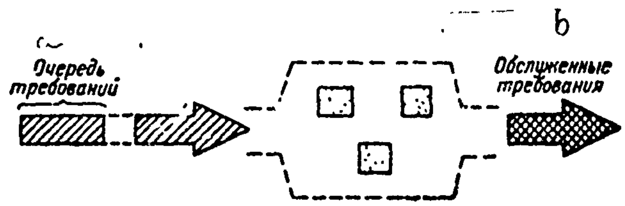

Fig. 27 - A Service System With Waiting. The Productive Capacity of the System is Insufficient; the Demands Can Wait.

Key: a - Line of demands
b - Serviced demands

A number of other criteria are of interest, in particular, the average waiting time before service begins. Here, the criterion of fullness of load becomes secondary, or auxiliary.

In other cases, (Fig. 28) a demand may not wait for service to begin (for instance, enemy planes flying into our defense lines), and if the productive capacity of the service system (Group of anti-aircraft complexes) is insufficient, then the output flow must consist of both serviced demands (aircraft shot down) and unserviced demands (planes breaking through).

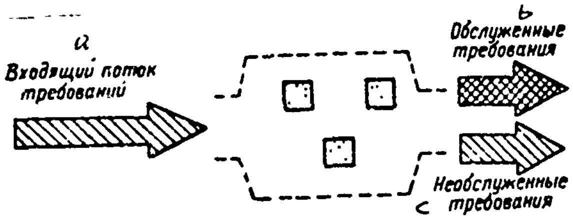

Fig. 28 - A Service System with Losses. The Productive Capacity of the System is Insufficient, the Demands Cannot Wait.

Key: a - Input flow of demands; b - Serviced demands; c - Unserviced demands.

Such a system is called a system with losses. The basic quantitative criterion of the quality of the work of such a system is the probability of refusal of service, i. e., the probability that, at the moment a demand arrives, all of the service apparatuses will be occupied with demands that have arrived earlier. Such a criterion, as the average number of apparatuses, occupied with service (i. e. the above-mentioned fullness of load of service apparatuses once again becomes secondary.

In practice, one meets other diverse systems of mass service, of a more complex character. Special literature is devoted to these systems, which we will not be concerned with here.

The Necessity of Applying the Theory of Mass Service

When the time interval between arrivals of demands for service is steady, and the time of servicing one demand is also steady, then no problem, studied by the theory of mass service, will arise. In this case, it is easy to find the needed number of service apparatuses. For example, if four tanks were to appear every minute in an anti-tank defense zone, and one anti-tank complex took 0. 5 min. to destroy one tank. then we would need 4 x 0. 5 = 2 complexes, so that not a single enemy tank would remain undestroyed ("unserviced").

The theory of mass service is needed in cases where the interval of time between arriving demands, or the time to service one

demand, or both of these quantities, are not steady, but oscillate within some boundaries around an average value.

As a rule, in the majority of military problems, the input flow of demands does not depend on our will, but on a number of chance factors, and in particular, the intentions of the enemy. Therefore, to determine the number of demands, arriving over a certain interval of time, one must resort to the probability characteristics of the input flow.

The time it takes to service one demand with one service apparatus is also, in the general case, an unsteady quantity. In fact, each of the planes, arriving in a repair network, will need a different amount of work done to it, depending on the nature of the damage. The firing cycle of an anti-aircraft complex can also vary, depending on the nature, altitude, speed, and parameters of an aerial target. Therefore, for each process of mass service, a law of service time distribution must be found, i. e. a function that will, for each interval of time, determine the probability that the service will be completed in this time interval.

The Probability Characteristics of An Input Flow of Demands

For many actual processes, the flow of demands is sufficiently accurately characterized by Poisson's Law of Distribution, according to which, the probability of exactly k demands arriving at a service system over a time interval t, is determined by the equation

$$P_{k(t)} = \frac{(\lambda t)^k}{k!} e^{-\lambda t},$$

where $P_k(t)$ is the probability of k demands arriving over time t,

λ is the density of the flow (the average number of demands arriving in a unit of time,

t is the time,

k is the number of demands in time t

e is the base of the natural logarithms[1],

! is the factorial sign[2].

In making a mathematical description of the mathematical processes, one must consider that reality is always richer than the principle, with whose aid we are trying to reflect it. Therefore, a distribution law reflects principles, not inherent in an input flow of demands actually met in practice, but in a type of input flow, defined by certain boundaries.

So, it is with Poisson's Distribution. In order to apply the principles of Poisson's Distribution to an actual input flow of demands, the actual flow must be contained in the following framework.

In the first place, the number of demands, making up the flow, must have only integral values (including zero).

In the second place, the number of demands, arriving for service in a given time segment, must not depend on the number of demands arriving at some other time segment. In other words, the arrival of demands at a given moment of time must not be connected with the arrival or non-arrival of demands at other moments of time.

In the third place, the arrival of two or more demands simultaneously must be practically excluded.

In the fourth place, the arrival of demands must be distributed in time in a random fashion. In other words, the input flow of demands must be characterized by a stable average number of demands arriving in equal (and sufficiently large) time intervals.

This average number of demands, designated in the formula by λ is called the density of the flow.

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[1]The base of the natural logarithms is $e = 2.718$. To ease calculations, a table is given in the end of the book of values of e^{-x} for different values of x (Appendix 1).

[2]The symbol $k!$, called the k-factorial, is the product of all intergral numbers from 1 to k, i.e. $1 \cdot 2 \cdot 3 \ldots (k-1)k$. Thus, for instance, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. At the end of the book there is a table of different values of $k!$ for different values of k (Appendix 2).

In spite of the seemingly rigid conditions, there are many real input demand flows, for which Poisson's Distribution is applicable, and this enables it to be used for practical calculations.

A flow of demands, which is subject to Poisson's law, is sometimes called the "simplest".[1]

Thus, for instance, for an attack density of one plane per minute (λ = 1), the probability of different numbers of planes passing the given zone per minute is given in the following table.

a Число самолетов в минуту, k	0	1	2	3	4	5	6	7
b Вероятность появления этого числа самолетов, $P_{k\,(1)}$	0,3679	0,3679	0,1839	0,0613	0,0153	0,0031	0,0005	0,0000

Key: a - Number of planes per minute, k
b - Probability of this number appearing, $P_{k(1)}$

From the table it is evident that the expression "density of attack equal to one plane per minute" cannot be interpreted as if every minute one plane will pass through. One plane per minute is only the average. There will be minutes when no planes fly into the area (almost 37% of all cases). And there will be minutes when two (18% of all cases), three (6% of all cases), or more airplanes enter the area.

Graphically, this distribution appears in the following form (Fig. 29).

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[1] If a distribution law is obtained by sorting the statistics of some flow of demands met in practice, then one of the indications that this distribution is subject to Poisson's Law is even the most approximate equalization of the mathematical expectation of the chance quantity and the dispersion of this quantity (concerning methods of determining the mathematical expectation and the dispersion of a chance quantity see Chapter II).

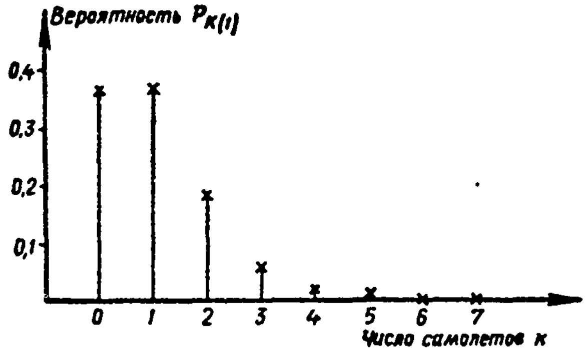

Fig. 29 - Law of Distribution of a Flow of Demands

Key: a - Probability $P_{k(1)}$; b - Number of airplanes k;

From the table and the graph, it is apparent that the passage of four or more airplanes per minute is practically impossible, (the probability of this event is so small, that it can be ignored), but we must consider the passage of two or three planes per minute in organizing against the assault, since this can happen approximately one time out of four.

The Probability Characteristic of Service Time

For many random processes met in practice, the law of service time distribution is sufficiently accurately described by an exponential function

$$F_{(t)} = 1 - e^{-\beta t} *,$$

where $F_{(t)}$ is the probability of completing the service in time t,

$1/\beta$ is the mathematical expectation of the service time (average time of service) of one demand by one service apparatus, **

e is the base of the natural logarithms.

---

* To ease practical calculations of the quantity $e^{-\beta t}$, the table of values of e^{-x}, at the end of the book, may serve;

** If $1/\beta$ is the average service time for one demand, then the inverse quantity β is the average number of demands, served in one unit of time by one service apparatus.

Thus, for instance, if the average time of servicing one demand by one apparatus is equal to 1 minute, the probabilities of completing the service in different time intervals are defined in the following table.

Интервал времени t, мин	0	1	2	3	4	5	6
Вероятность окончания обслуживания за время t $(F_{(t)})$	0	0,6321	0,8647	0,9502	0,9817	0,9933	0,9975

Key: a - Time interval t, min.
b - Probability of completing service in time t $(F_{(t)})$.

Graphically, this distribution appears in the following form.

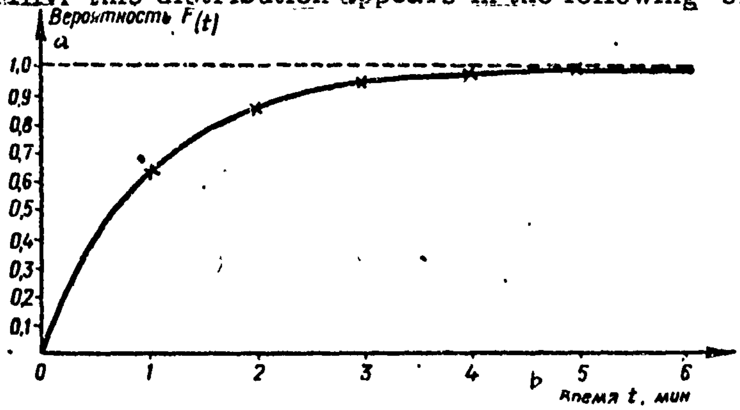

Fig. 30 - The Law of Distribution of Service Time.

Key: a - Probability $F_{(t)}$; b - Time t, min.

In general, there is sufficient reason to consider that the real process of service is subject to the exponential law, if the probability of completing the service soon after it has begun is high, and it is not probable that the service will extend over a protracted length of time.

Several Military Problems as Problems of the Theory of Mass Service

If the input flow of demands and the time to service one demand do not depend on us, then organizing the functioning of a service system, particularly the choice of the number of service apparatuses, is entirely up to our competition. What does it mean to organize a service system well? This means that a long line of demands does not form, that unserviced demands are not left over, and that our service apparatuses are not idle. It is necessary, first of all, to dispatch a corresponding number of service apparatuses (for instance, to correctly determine the array of forces, participating in a defense), to correctly group the service apparatuses and to distribute the demands among them in the most reasonable way (i. e. to make the most rational distribution of targets among defense weapons).

The value of the theory of mass service is increased by the fact that it gives us a means of quantitatively evaluating the quality of service, which is expressed by quantities characterizing the input flow of demands and the service time.

In the military, the theory of mass service can be most applied in the following problems:

-- problems of organizing various kinds of service systems, with the goal of constructing the optimum system for each concrete case; in this respect we have, first of all, evaluating the quality of defense systems, and also of repair, supply and medical-sanitary systems;

-- problems of organizing the guidance of forces in battle; in this respect we have, first of all, evaluating the quality of systems of communicating and revising information about a situation, which is especially important in creating automatic guidance systems;

-- problems associated with forming models of the processes of military operations.

For instance, in organizing a PVO (Protivovozdushnaya Oborona - Anti-aircraft Defense) or PRO (Protivoroketnaya Oborona - Anti-rocket defense), the process for firing at bombers (rockets) can be considered as a process of mass service, and the arrival of target data (points) on a radar screen can be seen as the arrival of demands

for service. Using methods of the theory of mass service, one may, in this case, calculate, for example, a criterion of effectiveness, such as the number (percent) of bombers (rockets) not destroyed by a given defense organization, which is the essential element in evaluating the combat capability of a given anti-aircraft rocket group. The theory of mass service will also help us to calculate the number of rocket complexes needed, so that with their given quality maintained, the aerial space or objective will be properly defended, and there will be no extra complexes.

An Example of a System of Mass Service with Losses.

We will examine a conditional, numerical example. Suppose we want to create a defense system, with the condition that the probability of breaking through this system is not more (in a given direction) than 0.02, i.e., that less than 2 out of 100 targets would penetrate the system. It is necessary to determine from what number of defense complexes the system must consist. It is obvious that the defense system can be related to systems of mass service with losses, since undestroyed targets will leave the defense zone without having been serviced. Therefore, the probability of breaking through the system is equivalent to the probability of denial of service. Consequently, the quality of the defense can be characterized by the probability of denial of service and the completeness of the load of the defense complexes in the course of repelling the targets. We will calculate these quantities.

Let us suppose that the flow of enemy targets in a given direction is simplest with a density of $\lambda = 4$ (the average number of targets reaching the defense zone in a unit of time is four) and the average time to fire at one target with one complex is $1/\beta = 1$ (on the average, one complex can fire at one target in one unit of time). We will also consider that the probability of destroying an enemy target in one round (volley) is equal to one (i.e. firing at a target is equivalent to destroying it).

The probability of breaking through some defense system is equal to the probability that all of the complexes of this system will happen to be busy (will be firing at other targets).

The needed number of defense complexes may then be found from an inequality, which stipulates that the probability of all the

complexes of this system will happen to be busy (will be firing at other targets).

The needed number of defense complexes may then be found from an inequality, which stipulates that the probability of all the complexes being occupied is not more than 0.02.

For a system of mass service with losses, the probability of all complexes being occupied is equal to [1]

$$P_{a} = \frac{\left(\frac{\lambda}{\beta}\right)^{r} \frac{1}{r!}}{\sum_{m=0}^{r} \left(\frac{\lambda}{\beta}\right)^{m} \frac{1}{m!}},$$

Key: a - occ

where P_{occ} is the probability that all complexes will be occupied,

λ is the density of the target flow

$1/\beta$ is the average time for one complex to fire at one target,

r is the number of defense complexes,

m is a running parameter, taking values from 0 to r.

Since we want the probability of all complexes being occupied to be less than 0.02, i.e. so that the condition

$$P_{a} < 0{,}02,$$

Key: a - occ

[1] This formula was obtained by Erlang, and is named after him.

is fulfilled, we must find a number of complexes r for which the above expression is less than 0.02.

To find the quantity r, we make a graph on whose abscissa we place the number of complexes and on whose ordinate we place the probability of penetrating[2] the defense system, calculated for this number of complexes, (Fig. 31).

The graph of P_{occ}, as one might expect, shows that as the number of complexes in a defense system increases, the probability of penetrating this system decreases. At first, this decrease is sharp, but then it becomes shallower and shallower.

If we place two complexes against a flow of targets of four per minute, and each of these complexes is capable, on the average, of repelling one target per minute, then the probability of both complexes being occupied simultaneously will equal to almost 60%. This means, that about 60% of the targets will not be destroyed.

Adding one more complex (making $r = 3$), will decrease the probability of penetration to 43%, i.e. by 17%.

With six complexes, the probability of penetrating the system will be equal to 12%, and the addition of one more complex ($r = 7$) will decrease it to 6%, i.e. by 6%.

Naturally, for an assault with a different density, the probability of penetrating this same system will be notably different. Thus, if the density is twice as high (eight targets per minute instead of four), the probability of penetration for a system of two complexes will rise to 78% (as opposed to 60%), and the probability of penetrating a system of six complexes will rise to 39% (as opposed to 11%).

This visually affirms the assertion that the reliability of a defense system cannot be judged without regard to the nature of the penetration of the enemy, who is trying to overcome it. If the system is sufficiently reliable for one variant of enemy activity, it may still turn out to be unsatisfactory for another variant. Therefore, a reliably constructed system must be calculated for the worst variant.

Returning to our example, we determine by the graph that, in the given case, in order to obtain a probability of penetration of less than

[2] The probability of penetration is equal to the probability of all complexes being occupied P_{occ}.

0.02, the number of complexes must not be less than nine.

It would seem that simple reasoning would lead to a different conclusion. In fact, fi, for instance, four enemy targets appear in a minute, and each complex spends one minute firing at each one, then it would be enough to have, in all, four complexes, so that not a single target would penetrate. Why then, does the theory of mass service demand that we establish nine complexes so that not more than two out of 100 targets will penetrate? This is because a density of = 4 indicates that four planes per minute is only the average. If, for instance, in the first minute five targets arrive, in the second - three targets, in the third - six, and in the fourth - two, i. e. in all 16 targets in 4 min., then, although on the average there are 16/4 = 4 targets per minute, with only four complexes there will be one target allowed through in the first minute and two in the third, which is a total of three out of 16. This means that, for this reason alone, the probability of penetration i four minutes will be high (about 19%), and the demand was that the probability be less than 2%.

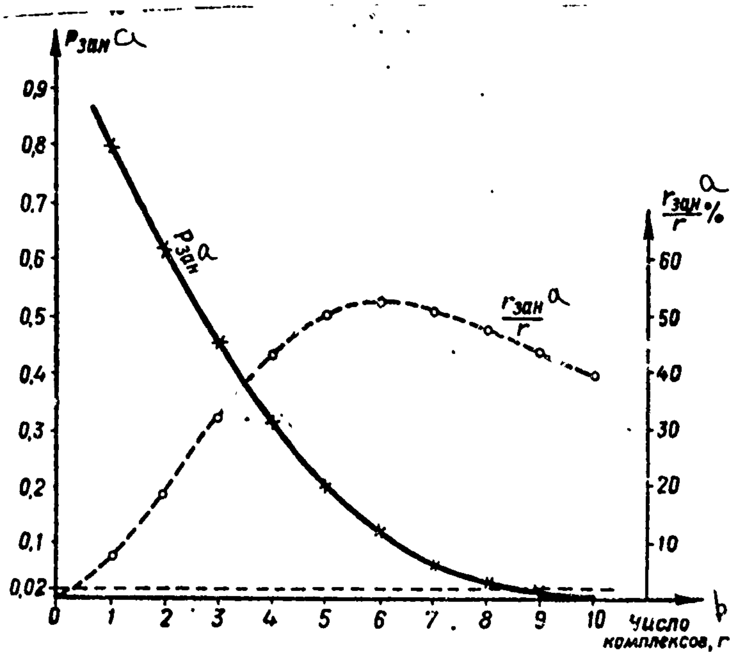

Fig. 31 - The Probability of all the Complexes Being Occupied Simultaneously and the Average Percent of Occupied Complexes

Key: a - occ; b - Number of complexes.

One must also not fail to consider that firing at one target requires one minute, only as an average. This time can actually turn out to be more or less than a minute.

When considering all of these vaccilations of flow density and time of service, we must have nine complexes to keep the probability of penetration from exceeding 2%. With only four complexes, an average of 30% of the targets will be let through (as is apparent from the graph), i. e. approximately every third target.

In the given example we see that when a process is of a random nature, simple reasoning can lead to a rather serious error.

Calculations which are not shown here, show that the average number of occupied complexes at each moment in the reflection of an assault will be four (out of nine). This means that each complex will be occupied $4/9 \times 100 = 44.5\%$ of the time of the assault. If there are four launching pads in all, the average number of occupied complexes will be 1.74. Consequently, each launching pad will be busy $1.74/4 \times 100 = 43.5\%$ of the time of the assault. Therefore, with nine launching pads the defense of the aerial space is increased, and the fullness of load on the launching pads does not decrease.

In Fig. 31 a dotted line shows the fullness of load on the complexes, for different numbers of them, pertaining to the example we have just analysed (here, by fullness of load we mean the ratio of the average number of occupied complexes to their overall number, expressed in percentages). From the graph it is apparent, that the greatest fullness of load is achieved when $r = 6$ complexes (52.3% of the time). However, here the probability of all six complexes being occupied simultaneously is $P_{occ} = 0.12$ (i. e. 12 targets out of 100 may penetrate). Obviously, the chief criterion in the given problem is the realibility of the defense of the aerial space. The criterion of occupation can only be secondary.

An Example of a System of Mass Service with Waiting

We shall analyze another example. Suppose our staff is given the problem of dispatching duty equipment of a certain type to

destroy targets that have just been discovered by aerial reconnaisance. The question arises: "How many units should be sent?"

From the viewpoint of the theory of mass service, each duty device can be seen as a service apparatus, and the total of these as a service system. In this case, the demands for service are reports of newly discovered targets, whose destruction is decided by the duty devices. The process of service itself will consist of firing at the targets.

What may be saidin the given example, about the number of demands arriving in a unit of time? Obviously, it will not be constant. In certain hours, orders to fire at targets will come more frequently, at other hours more rarely, and at other hours, none at all.

However, on the basis of the preceding experiment it may be established that in given meteorological conditions, at a given time of day, for a given number and type of reconnaisance, and observation regions of the same dimensions, one may expect, on the average, a certain number of commands per hour. Suppose the mathematical expectation of the number of commands per hour is $\lambda = 8$.

What may be said about the time to service one demand with one service apparatus in the given example? Obviously, this will also not be constant. The time spent by one device in shooting at one target will consist of the time to transmit the coordinates, the time to make a decision, the time to transmit the commands, the time to prepare the initial launching data, and the time to strike against the target. Each of these time segments will have a different value in an actual situation.

However, on the basis of the preceding experiment, one may evaluate the average time to use one device on one target. Let the mathematical expectation of this time be equal to $1/\beta = 1/2$ hr. (where $\beta = 2$, i. e. one duty unit can destroy, on the average, two targets in 1 hour)

Consequently, the problem of determining the needed number of units is a problem, which demands a solution by the methods of the theory theory of mass service. Unlike the preceding example, where we were dealing with a so-called system with losses, (the demands could not wait for service to begin), in this example we are studying a system

with waiting (the demands can wait for service to begin[1], a line of demands may accumulate.

To determine the quantity of duty equipment we will attempt at first to use simple reasoning. In fact, if one duty unit can fire at two targets in one hour, and in the course of one hour information arrives concerning eight targets, then it would seem that four duty units would be enough ($r = 8/2 = 2$), so that all of the newly discovered targets would be fired upon on time.

However, as in the first example, one must not forget that the arrival of information of eight targets per hour is only an average. In certain hours more information may come, and there will not be enough units, and in other hours less data will arrive and units will be wasted.

In particular, if we consider, that the flow of demands in the given case is subject to Poisson's Law, we may find that the probability of exactly $\lambda = 8$ demands arriving in the course of an hour is not very high. It will be equal to

$$P_{k(t)} = \frac{(\lambda t)^k}{k!} e^{-\lambda t},$$

$$\therefore \quad P_{8(1)} = \frac{(8 \cdot 1)^8}{8!} e^{-8 \cdot 1} = \frac{8^8}{8!} e^{-8} \approx 0{,}14,$$

i. e. exactly eight demands will arrive in only 14% of the cases. If daylight lasts, for instance, 15 hours of the day, then exactly eight demands will arrive in two of those hours. In the remaining 13 hours, fewer or more demands will arrive.

One must also not forget that one unit can fire at one target for half an hour only as an average. More time may be expended on some targets, and less on others.

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[1] However, they cannot wait for service more than a definite amount of time, upon whose expiration the target will change its coordinates, will have to be resighted and placed in line for service once again.

In particular, if we consider that the time of service is subject to the exponential law, we may find that the probability of completing our firing in one half hour is not very large. It will be equal to

$$F_{(t)} = 1 - e^{-\beta t},$$

from which

$$F_{(1/2)} = 1 - e^{-2 \cdot \frac{1}{2}} = 1 - e^{-1} \approx 0{,}63,$$

i.e. in 37% of the cases firing at one target will occupy more than half an hour. Therefore, simple reasoning, as in the first example, has lead to a mistake.

In particular, for a service system with waiting, we may say in advance that if the number of service apparatuses is less than the ratio λ/β or equal to it, i.e. if

$$r < \frac{\lambda}{\beta},$$

then the line of unserviced demands (and consequently, the time of waiting for service to begin) will grow without limit.

Therefore, the first step in checking such systems is to determine the ratio λ/β. In our example $\lambda/\beta = 8/2 = 4$, from which we may draw the conclusion that the quantity of duty equipment must be more than four. In some cases, we must stipulate that a target must be fired upon not less than 30 minutes after its discovery. In other cases, we may be satisfied with 2 or 3 hours, 24 hours, etc. This depends on the character of the target and conditions that complicate the situation.

Returning to the solution of our example of finding the needed number of devices, and setting up five of them ($r = 5$), we study the characteristics of this service system with waiting.[1]

[1] Once again a reminder is made that the formulas introduced below are correct if $r > \lambda/\beta$ (in our example they can be used as long as the number of batteries is $r > 4$.

First of all, we must find P_0, the probability that when the next in line command arrives, all five units will be free.

$$P_0 = \frac{1}{\sum_{m=0}^{r-1} \frac{1}{m!}\left(\frac{\lambda}{\beta}\right)^m + \frac{\beta}{(r-1)!(r\beta-\lambda)}\left(\frac{\lambda}{\beta}\right)^r},$$

where m is a running parameter, acquiring integral values from 0 to r - 1, and the remaining symbols are clear from the preceding examples.

In our example r = 5, λ = 8, β = 2, therefore

$$P_0 = \frac{1}{\sum_{m=0}^{4} \frac{1}{m!} 4^m + \frac{2 \cdot 4^5}{4! \cdot 2}} \approx 0{,}013 = 1{,}3\%.$$

Then we will find P_r, the probability that at the moment the next in line command arrives, all five units will be occupied.

$$P_r = \frac{\beta\left(\frac{\lambda}{\beta}\right)^r}{(r-1)!(r\beta-\lambda)} P_0,$$

from which

$$P_r = \frac{2 \cdot 4^5}{4!(5 \cdot 2 - 8)} 0{,}013 \approx 0{,}55 = 55\%.$$

Now we calculate $P_{\tau > t}$, the probability that the next target after it will have to wait more than t hours for service:

$$P_{\tau>t} = P_r e^{-(r\beta-\lambda)t}.$$

Assigning a value to t of 0, 1/2, 1, and 2 hrs. we may, with the aid of this formula, find a law of waiting time distribution

t h	0	$1/2$	1	2
$P_{\tau>t}$	0,55	0,2	0,075	0,01

Graphically, this law will appear in the following form

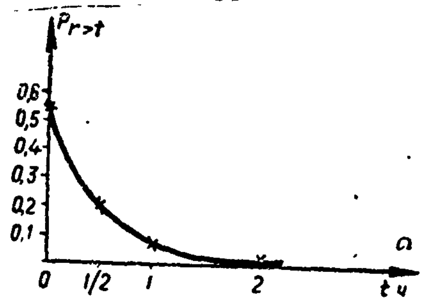

Fig. 32 - Law of Waiting Time Distribution

Key: a - hour

Therefore, the probability that a target will be fired at, later than one half hour after discovery, is equal to 0. 2, i. e. this will occur in every five targets.

Thus, the average time of waiting for service to begin will be equal to

$$T = \frac{P_r}{r\beta - \lambda},$$

from which

$$T = \frac{0{,}55}{5\cdot 2 - 8} \approx 0{,}275 \overset{a}{\text{ч}} = 16{,}5 \overset{b}{\text{мин}}^{1}.$$

Key: a - hour; b - min.[1]

Finally, the average number of units, not occupied in firing, will be equal to

$$R = \sum_{m=0}^{r-1} \frac{r-m}{m!} \left(\frac{\lambda}{\beta}\right)^m P_0,$$

from which

$$R = \sum_{m=0}^{4} \frac{5-m}{m!} \cdot 4^m \cdot 0{,}013 \approx 1.$$

---

[1] In particular, the probability that the waiting time will be more than the average waiting time will be equal to

$$P_{\tau > 0{,}275} = 0{,}55 e^{-2\cdot 0{,}275} \approx 0{,}32 = 32\%.$$

Consequently on the average one unit will free and four will be occupied, therefore, the fullness of use for each unit will be $4/5 \cdot 100 = 80\%$. If it is light for 15 hours, then each unit will be occupied for 12 hours of that time.

These calculations show that if five units are designated for duty, then on the one hand newly discovered targets will be fired upon soon enough (in 16.5 min. on the average), and on the other hand, there will be a sufficiently high load on these units (80%).

* * * * * * * * * *

In many mass service problems, met in practice, the input flow is simplest and the service time is completely subject to a well defined law (in particular, the exponential law).

However, in other practical problems, the input flow may be anything but the simplest; the law of service time distribution may turn out to be arbitrary, and the organization itself may be of a complicated, multiphases nature.

Developed analytical methods still do not exist for all cases, but is is possible for us to obtain interesting quantitative characteristics of service quality by modeling the service processes on electronic computers.

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